MATHEMATICS

GRADE 12

STUDENT TEXTBOOK

CONTENTS

MATHEMATICS12

SEMESTER–I

SEMESTER–II

SEMESTER–I

PERIOD–I

M12CH1

TOPIC

1

Sequence and Series

1.1. SEQUENCES

A succession of numbers formed according to a certain rule and arranged in a definite order is called a **sequence**.

Illustration. $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots, \frac{1}{2n} \ldots$ is a sequence.

In a sequence, the numbers occurring at its first place, second place, third place, ... nth place are respectively called its first term, second term, third term, ..., nth term.

1.2. ARITHMETIC PROGRESSIONS

A sequence is said to be an **arithmetic progression (AP)** if the difference of each term, except the first one, from its preceding term is always same.

For example 2, 5, 8, 11, ... is an AP, because

$$5 - 2 = 3,\ 8 - 5 = 3,\ 11 - 8 = 3, \ldots$$

Thus, the sequence $\{T_n\}$ is an arithmetic progression, if there exists a number, say, d such that $T_{n+1} - T_n = d$ for $n \geq 1$.

Definition of an AP

If 'a' and 'd' be the first term and common difference of the AP $\{T_n\}$.

$$T_n = a + (n - 1)d, \quad n \in \mathbf{N}.$$

Example 1. *Find the number of terms in the Arithmetic progression (A.P.) 7, 10, 13, ..., 31.*

Solution. Given A.P. is 7, 10, 13, ..., 31

Since first term $a = 7$, common difference $d = 10 - 7 = 3$ and $T_n = 31$.

We know that $T_n = a + (n-1)d$

$$31 = 7 + (n-1)3$$

$$31 - 7 = 3n - 3$$

$$24 = 3n - 3$$

$$3n = 27$$

$$n = 9$$

Hence, the number of terms in the A.P. is 9.

Example 2. *Find the sum of the following arithmetic progression 9, 15, 21, 27, ... The total number of term is 14.*

Solution. Given A.P. is 9, 15, 21, 27, ...

Here first term $a = 9$, common difference $d = 15 - 9 = 6$

Number of term $n = 14$.

We know that $S_n = \frac{n}{2}[2a + (n-1)d$

$$S_{14} = \frac{14}{2}[2 \times 9 + (14-1)6]$$

$$= 7[18 + 78] = 7 \times 96 = 672$$

Hence, the sum of the given A.P. is 672.

Example 3. *Find the m^{th} term of an A.P. sum of whose first n terms is $2n + 3n^2$.*

Solution. Given that sum of first n terms of an A.P. is $2n + 3n^2$

i.e., $S_n = 2n + 3n^2$

Hence the m^{th} term of the A.P. is

$$T_m = S_m - S_{m-1}$$

$$= (2m + 3m^2) - \{2(m-1) + 3(m-1)^2\}$$

$$= (2m + 3m^2) - \{(2m-2) + 3(m^2 + 1 - 2m)\}$$

$$= (2m + 3m^2) - (2m - 2 + 3m^2 + 3 - 6m)$$

$$= 2m + 3m^2 - 2m + 2 - 3m^2 - 3 + 6m$$

$$= 6m - 1.$$

Example 4. *Find the 17^{th} term from the end of the A.P. 1, 6, 11, 16, ..., 21, 216.*

Solution. Given A.P. is 1, 6, 11, 16, ..., 211, 216.

Here $a = 1$, $d = 6 - 1 = 5$, and last term $l = 216$, $n = 17$

We know that: $T_n = l - (n-1)d$

$T_{17} = 216 - (17 - 1)5$

$= 216 - 80 = 136.$

1.3. GEOMETRIC PROGRESSIONS

A sequence of *non-zero* numbers is said to be a **geometric progression** (**GP**) if the ratio of each term, except the first one, by its preceding term is always same.

For example, 3, 6, 12, 24, ... is a GP, because

$$\frac{6}{3} = 2, \frac{12}{6} = 2, \frac{24}{12} = 2, \cdots$$

Thus, the sequence $\{T_n\}$ with $T_n \neq 0$ is a geometric progression if there exists a non-zero number, say, r such that $\frac{T_{n+1}}{T_n} = r$ for $n \geq 1$.

The constant number 'r' mentioned above is called the **common ratio** of the corresponding GP. The common ratio of a GP is denoted by 'r'.

The first term of a GP, is generally denoted by 'a'.

Definition of a GP

If 'a' and 'r' be respectively the first term and common ratio of the GP $\{T_n\}$, then

$$T_n = ar^{n-1}, \quad n \in \mathbf{N}. \qquad ...(1)$$

Example 5. *Find the 9th and nth terms of the sequence 3, 6, 12, 24, ...*

Solution. Given sequence is 3, 6, 12, 24, ...(1)

Here, $\frac{T_2}{T_1} = \frac{6}{3} = 2, \quad \frac{T_3}{T_2} = \frac{12}{6} = 2, \ldots \qquad \therefore \quad \frac{T_2}{T_1} = \frac{T_3}{T_2} = \ldots = 2$

$\therefore$ (1) is a GP with $a = 3$ and $r = 2$.

Now, $T_9 = ar^{9-1} = ar^8 = 3(2)^8 = 3(256) = \mathbf{768}$ $\qquad [T_n = ar^{n-1}]$

and $T_n = ar^{n-1} = \mathbf{3(2)^{n-1}}$.

Example 6. *Which term of the series $\frac{1}{4} - \frac{1}{2} + 1 + \ldots$ is 256?*

Solution. The series is $\frac{1}{4} + \left(-\frac{1}{2}\right) + 1 + \ldots$

Here, $\frac{T_2}{T_1} = \frac{-1/2}{1/4} = -2, \quad \frac{T_3}{T_2} = \frac{1}{-1/2} = -2, \ldots$

$\therefore \quad \frac{T_2}{T_1} = \frac{T_3}{T_2} = \ldots = -2$

$\therefore$ The given series is a GS with $a = 1/4$ and $r = -2$.

Let $T_n = 256 \qquad \therefore \quad ar^{n-1} = 256$

$\Rightarrow \quad = 256 \qquad \Rightarrow \quad (-2)^{n-1} = 1024$

$\Rightarrow \quad (-2)^{n-1} = 210 \qquad \Rightarrow \quad (-2)^{n-1} = (-2)^{10}$

$\Rightarrow \quad n - 1 = 10 \qquad \Rightarrow \quad n = 11.$

$\therefore$ 256 is the **11th term.**

Example 7. *For what value of n, the nth terms of the series "5 + 10 + 20 + ..." and "1280 + 640 + 320 + ..." are equal?*

Solution. Ist series: We have 5 + 10 + 20 + ...

Here, $a_1 = 5$ and $r_1 = \frac{10}{5} = \frac{20}{10} = 2$

$\therefore \quad T_n = a_1 r_1^{\,n-1} = 5(2)^{n-1}$

IInd series: We have 1280 + 640 + 320 + ...

Here, $a_2 = 1280$ and $r_2 = \frac{640}{1280} = \frac{320}{640} = \frac{1}{2}$

$\therefore \quad T_n = a_2 r_2^{\,n-1} = 1280 \left(\frac{1}{2}\right)^{n-1}$

Sum of first *n* Terms of A GP

The sum of first n terms of a GP is denoted by S_n.

If $\{T_n\}$ is a GP, then we have

$$S_n = T_1 + T_2 + T_3 + \ldots + T_n, \; n \in \mathbf{N}.$$

For example, 2, 6, 18, 54, L is a GP and:

$$S_1 = 2, \; S_2 = 2 + 6 = 8, \; S_3 = 2 + 6 + 18 = 26 \text{ etc.}$$

Definition

If a and r be respectively the first term and common ratio of a GP, then the sum of first n terms of this GP is given by

$$S_n = \begin{cases} na & \text{if} \quad r = 1 \\ \dfrac{a(1-r^n)}{1-r} & \text{if} \quad r \neq 1. \end{cases}$$

Example 8. *Find the 20th term of the G.P.* $\frac{5}{5}, \frac{5}{4}, \frac{5}{8}, \ldots\ldots$.

Solution. Given G.P. is $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \ldots\ldots$

Here first term $a = \frac{5}{2}$

Common ratio $r = \dfrac{\frac{5}{4}}{\frac{5}{2}} = \frac{5}{4} \times \frac{2}{5} = \frac{1}{2}$

and $n = 20$

We know that $T_n = ar^{n-1}$

$$T_{20} = \frac{5}{2}\left(\frac{1}{2}\right)^{20-1} = \frac{5}{2} \times \left(\frac{1}{2}\right)^{19} = 2^{\frac{5}{20}}.$$

Example 9. *In a G.P.* $T_{10} = 9$ *and* $T_4 = 4$, *then find* T_7.

Solution. We know that $T_n = ar^{n-1}$

$\therefore$ $T_{10} = 9 = ar^9 \Rightarrow ar^9 = 9$ ...(*i*)

$T_4 = 4 = ar^3 \Rightarrow ar^3 = 4$ ...(*ii*)

Multiply equation (*i*) and (*ii*), we get

$$(ar^9)\,(ar^3) = 9 \times 4$$
$$a^2r^{12} = 36$$
$$(ar^6)^2 = 36$$
$$ar^6 = 6$$

i.e., $T_7 = 6$ $[\because \; T_n = ar^{n-1}]$

Example 10. *Find the sum of 7 terms of the G.P. 3, 6, 12, ...*

Solution. Here $a = 3$, $r = \frac{6}{3} = 2$ and $n = 7$

We know that $S_n = a\left[\frac{r^n - 1}{r - 1}\right]$

$\left[\text{If } r < 1 \text{ then use this formula } i.e., a\left[\frac{1 - r^n}{1 - r}\right]\right]$

$$= 3\left[\frac{(2)^7 - 1}{2 - 1}\right] = \frac{3(128 - 1)}{1} = 3 \times 127 = 381.$$

EXERCISE

1. Find the 20th and nth term of the sequence 4, 9, 14, 19, ...

2. Show that the sequence: log a, $\log \frac{a^2}{b}$, $\log \frac{a^3}{b^2}$, ... is an AP.

3. Which term of the series 37 + 32 + 27 + 22 + ... is –103?

4. Determine the 1st term and the 40th term of the AP whose 7th term is 34 and 15th term is 74.

5. Find the sum of all 3-digit numbers which leave the remainder 1 when divided by 4.

6. Evaluate:

(i) $\frac{1}{9} + \frac{2}{9} + \frac{1}{3} + \ldots +$ 25 terms (ii) 5 + 13 + 21 + ... + 181.

7. How many terms of the sequence –12, –9, –6, –3, ... must be taken to make the sum 54?

8. The first term of a GP is 1. The sum of third and fifth terms is 90. Find the common ratio of the GP.

9. If the 5th term of a GP is 16 and the 10th term is 1/2, find the GP. Also, find its 15th term.

10. Determine the number of terms in the GP $\{T_n\}$ if $T_1 = 3$, $T_n = 96$ and $S_n = 189$.

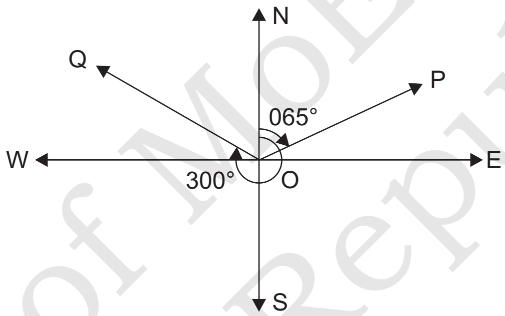
PERIOD-I

TOPIC 2

Bearing

2.1. DEFINITION

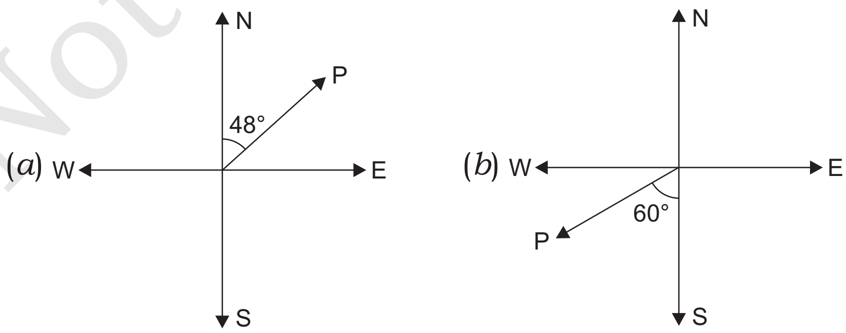
A tree bearing to a point is the angle measured in degree in a clockwise direction form the north line. We will refer to the tree bearing simply as the bearing.

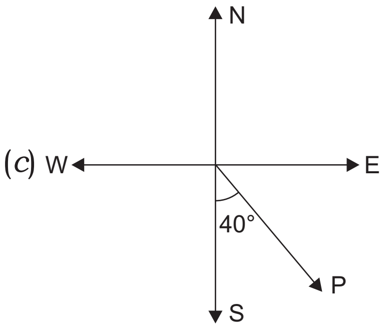


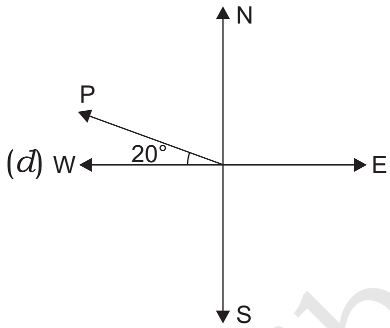
For example, the bearing of point P is 065° which is the number of degrees in the angle measured in a clockwise direction from the north line to the line joining the centre of the compass at O with the point P (*i.e.*, OP).

The bearing of point Q is 300° which is the number of degree in the angle measured in a clockwise direction from the north line to the line joining the centre of the compass at O with the point Q (*i.e.*, OQ).

Example 1. *State the bearing of the point P in each of the following diagrams:*

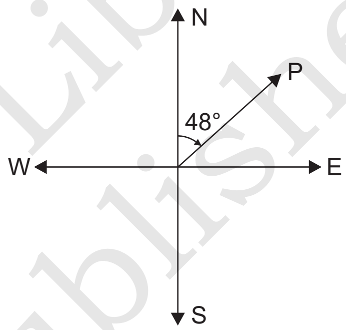






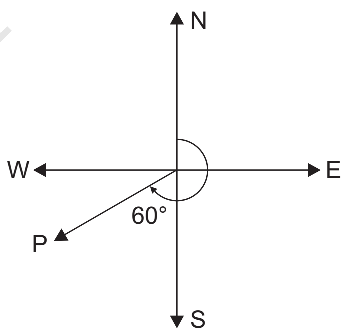
Solution. (*a*) Mark the angle in a clockwise direction by indicating the turn between the north line and the line joining the centre of the compass to the point P.

The bearing of point P is 048°.



(*b*) Mark the angle in a clockwise direction by indicating the turn between the north line and the line joining the centre of the compass to the point P.

The cardinal point S corresponds to 180°. It is clear from the diagram that the required angle is 60° larger than 180°. So the angle we asured in a clockwise direction from the north line joining the centre of the compass to point P is 180° + 60° = 240°.

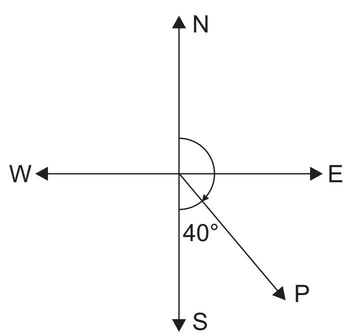


So, the bearing of point P is 240°.

(*c*) Mark the angle in a clockwise direction by indicating the turn between the north line and the line joining the centre of the compass to the point P.

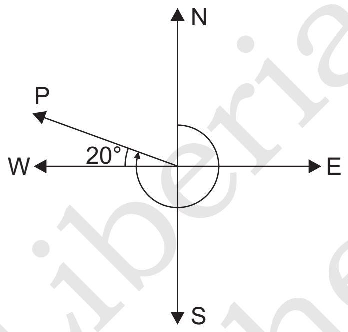
The cardinal point S corresponds to 180°. It is clear from the diagram that the required angle is 40° less than 180°.

So, the angle measured in a clockwise direction from the mark line to the line joining the centre of the compass to point P is 180° – 40° = 140°.



So, the bearing of point P is 140°.

(*d*) Mark the angle in a clockwise direction by indicating the turn between the north line and the line joining the centre of the compass to the point P.



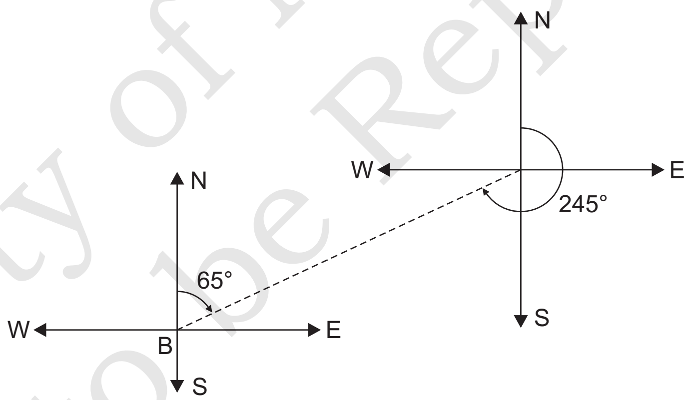
The cardinal point *w* corresponds to 270°. It is clear from the diagram that the required angle is 20° larger than 270. So, the angle measured in a clockwise direction from the north line to the line joining the centre of the compass to point P is 270° + 20° = 290°.

So, the bearing of point P is 290°.

Note: The bearing of a point is the number of degrees M the angle measured in a clockwise direction from the north line to the line joining the centre of the compass with the point.

A bearing is used to represent the direction of one point relative to another point.

For example, the bearing of A from B is 065°. The bearing of B from A is 245°.

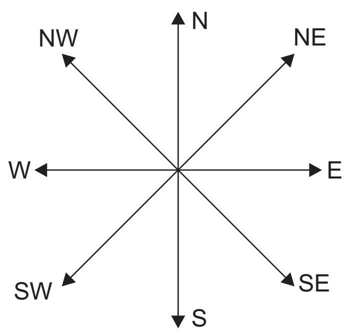


Example 2. *On what bearings is a ship sailing if it is heading*

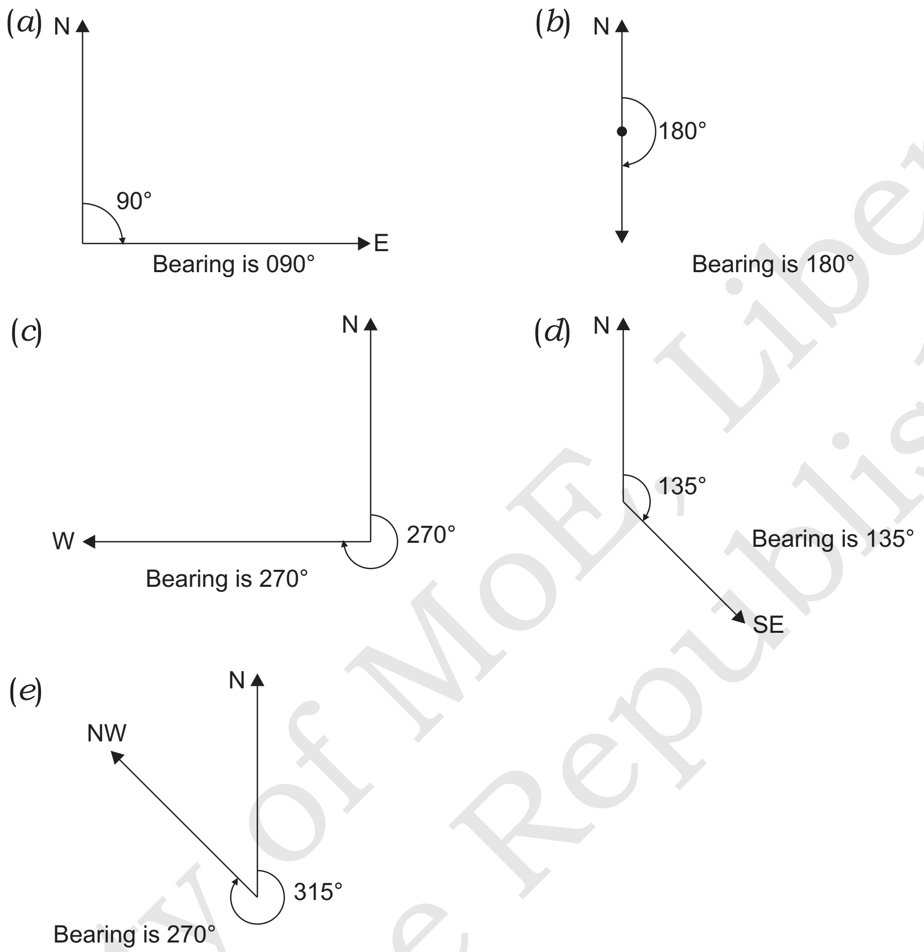
(*a*) *E* (*b*) *S*

(*c*) *W* (*d*) *SE*

(*e*) *NW*

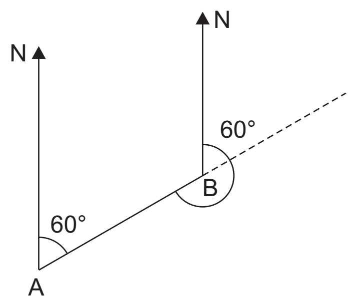


Solution.



Example 3. *A ship sails from A to B on a bearing of 060° on what bearing must it sail if it is to return from B to A?*

Solution. The diagram shows the journey from A to B extending the line of the journey allows an angle of 60° to be marked at B.



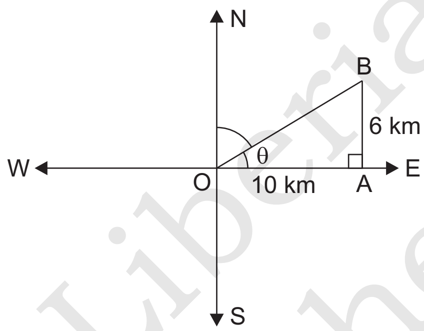
Bearing of A from B

$$= 60° + 180° = 240°$$

and this is called a back bearing or a reciprocal bearing.

Example 4. *A man walks 10 km east and then 6 km north. Find the bearing from the start to the finish point.*

Solution. Since the distances given are in compass directions of east and north, a right-angled triangle can be drawn in which the 10 km east is the adjacent side and the 6 km north is the opposite side



We have $\tan \theta = \frac{AB}{OA}$

$$\tan \theta = \frac{6}{10}$$

$$\tan \theta = \left(\frac{3}{5}\right)$$

$$\theta \cong 31°$$

So, the bearing from the start to the finish point

$$= 90° - 31° = 59°.$$

EXERCISE

1. If the bearing of Y from X is 060°, find the bearing of X from Y.
2. If the bearing of P from Q is 245°, find the bearing of Q from P.
3. If the bearing of a point Q from another point P is 040°, find the bearing of P from Q.

PERIOD–I

M12CH3

TOPIC **3**

Constructions

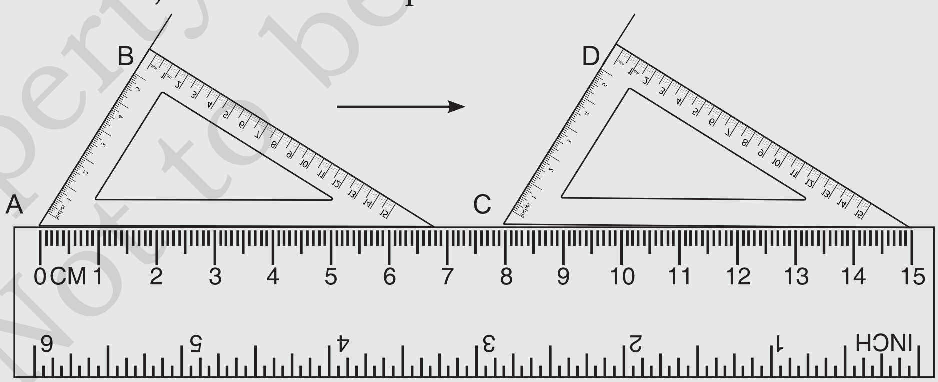
3.1. CONSTRUCTION WITHOUT MEASUREMENT

ACTIVITY 1

Aim: To draw parallel and perpendicular lines using a set square.

Materials Required: Set square, ruler, pencil.

- Divide the class in pairs.
- Instruction them to use a set square and ruler to draw parallel lines.
- Keep the set square on the ruler and draw $\overrightarrow{AB}$, now move the set square along the ruler as shown in figure.
- Draw $\overrightarrow{CD}$, observe $\overrightarrow{CD}$ is parallel to $\overrightarrow{AB}$

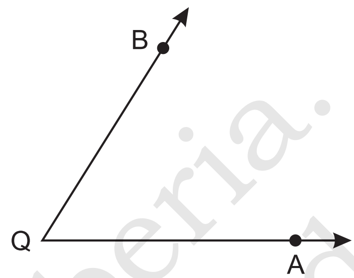


3.2. BISECTOR OF AN ANGLE

Bisecting an angle means drawing a ray in the interior of an angle such the angle is divided into two equal angles.

Construction of the Bisector of a Given Angle using a Pair of Compasses

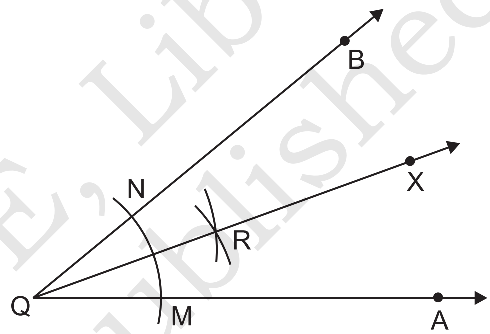
An angle ∠*AQB* is given. Let us draw a ray *OX*, bisecting ∠*AQB*.



Steps of Construction

Step 1: With Q as centre and any radius, draw an arc, cutting $\overrightarrow{QA}$ and $\overrightarrow{QB}$ at *M* and *N* respectively.

Step 2: With *M* as centre and radius more than half of *MN*, draw an arc. With centre at *N* and the same radius as before, draw another arc, cutting the previous drawn arc at a point *R*.



Step 3: Join *QR* and produce it to any point *X*.

Then, ray $\overrightarrow{QX}$ bisects ∠*AQB* into two equal angles, namely, ∠AQX, and ∠*BQX*.

3.3. CONSTRUCTION OF 75°, 105°, 135° AND 150°

(a) Let us first revise the construction of 30°, 45°, 60° and 90°.

(i) To construct an angle of 60°

Steps of construction

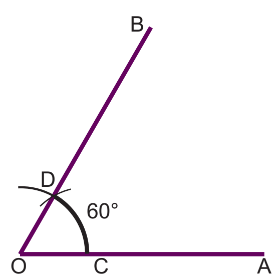
1. Draw any straight line *OA*.

2. With *O* as centre and any (suitable) radius, draw an arc to meet *OA* at *C*.

3. With *C* as centre and same radius as in step 2, draw an arc to meet the previous arc at *D*.

4. Join *O, D* and produce it to *B*.

Then ∠*AOB* = 60°.



(ii) To construct an angle of 30°

We observe that $\frac{0° + 60°}{2}$ = 30°, therefore, if ∠*AOB* = 60° and *OE* is the bisector of ∠*AOB,* then ∠*AOB* = 30°.

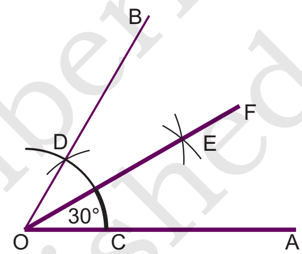
Steps of construction

1. Construct ∠*AOB* = 60°

2. With *C* as centre and any suitable radius (greater than half of *CD*), draw an arc. Also, with *D* as centre and same radius, draw another arc to meet the previous arc at *E*.

3. Join *O, E* and produce it to *F*. Then *OF* bisects ∠*AOB*.

Therefore, ∠*AOF* = 30°



(iii) To construct an angle of 90°

Steps of construction

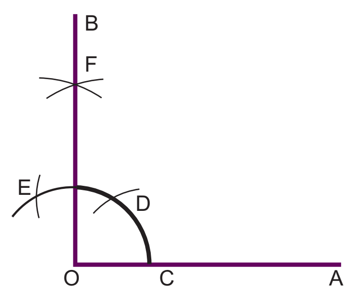
1. Draw any straight line *OA*.

2. With *O* as centre and any suitable radius, draw an arc to meet *OA* at *C*.

3. With *C* as centre and same radius, (as in step 2) draw an arc to meet the previous arc at *D*. With *D* as centre and same radius, draw another arc to meet the first arc at *E*.

4. With *D* and *E* as centres draw two arcs of equal radius (greater than half of *CD*), cutting each other at *F*.

5. Join *O, F* and produce it to *B*. Then ∠*AOB* = 90°



Note that: $\frac{60° + 120°}{2} = \frac{180°}{2} = 90°$

Here ∠ *AOD* = 60°,

$$\angle AOE = 120° \text{ so that}$$

$$\angle DOE = 120° - 60° = 60°$$

$$\angle DOF = \frac{1}{2} \times 60° = 30°,$$

because *OF* is the bisector of ∠*DOE*.

$$\angle AOF = \angle AOD + \angle DOF$$
$$= 60° + 30° = 90°$$

(iv) To construct an angle of 45°

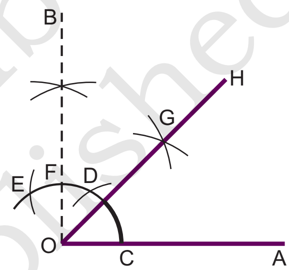
We observe that $\frac{0° + 90°}{2} = 45°$, therefore, if $\angle AOB = 90°$ and OG is the bisector of $\angle AOB$, then $\angle AOG = 45°$

Steps of construction

1. Construct $\angle AOB = 90°$

2. With C as centre and any suitable radius greater than $\frac{1}{2}CF$, draw an arc. Also, with F as centre and same radius, draw another arc to meet the previous arc at G.

Note that: $\frac{0° + 90°}{2} = \frac{90°}{2} = 45°$



Here OH is the bisector of $\angle AOB$,

3. Join O, G and produce it to H.

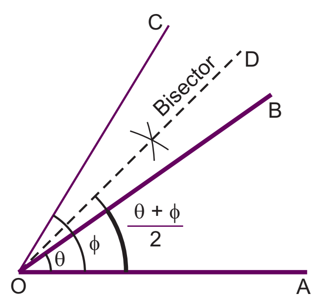
Then $\angle AOH = 45°$

(b) Now, let us construct angles of 75°, 105°, 135°, and 150°.

We observe that if $\angle AOB = \theta$ and $AOC = \phi$, then $\angle BOC = \angle AOC - \angle AOB = \phi - \theta$. If OD bisects $\angle BOC$, then

$$\angle BOD = \frac{1}{2}\angle BOC = \frac{\phi - \theta}{2} \text{ so that}$$

$$\angle AOD = \angle AOB + \angle BOD$$
$$= \theta + \frac{\phi - \theta}{2} = \frac{\theta + \phi}{2}.$$



(i) To construct an angle of 75°

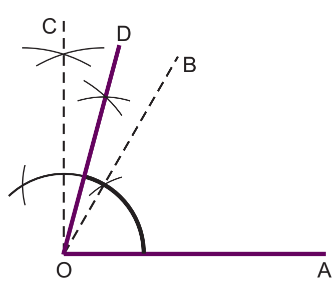
Note that: $\frac{60° + 90°}{2} = \frac{150°}{2} = 75°$

Steps of construction

1. Construct $\angle AOB = 60°$

2. Construct $\angle AOC = 90°$

then $\angle BOC = 90° - 60° = 30°$



3. Let *OD* be the bisector of $\angle BOC$, then $\angle BOD = \frac{1}{2} \times 30° = 15°$, so that

$$\angle AOD = \angle AOB + \angle BOD$$
$$= 60° + 15° = 75°$$

(ii) To construct an angle of 105°

Note that: $\frac{90° + 120°}{2} = \frac{210°}{2} = 105°$

Steps of construction

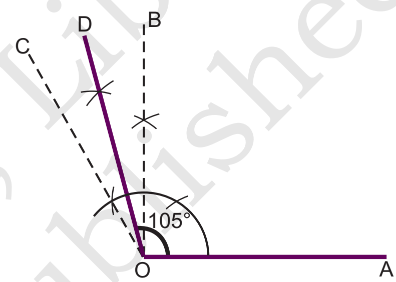
1. Construct $\angle AOB = 90°$

2. Construct $\angle AOC = 120°$

then $\angle BOC = 120° - 90° = 30°$

3. Let *OD* be the bisector of $\angle BOC$

then $\angle BOD = \frac{1}{2} \times 30° = 15°$, so that



$$\angle AOD = \angle AOB + \angle BOD$$
$$= 90° + 15° = 105°$$

(iii) To construct an angle of 135°

Note that: $\frac{90° + 180°}{2} = \frac{270°}{2} = 135°$

Steps of construction

1. Construct $\angle AOB = 90°$

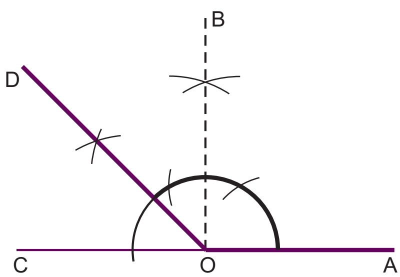
2. Produce *AO* to *C*, then

$\angle AOC = 180°$

and $\angle BOC = 180° - 90° = 90°$

3. Let *OD* be the bisector of $\angle BOC$

then $\angle BOD = \frac{1}{2} \times 90° = 45°$, so that

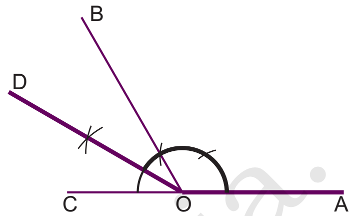


$$\angle AOD = \angle AOB + \angle BOD$$
$$= 90° + 45° = 135°$$

(iv) To construct an angle of 150°

Note that: $\frac{120° + 180°}{2} = \frac{300°}{2} = 150°$

Steps of construction



1. Construct $\angle AOB = 120°$

2. Produce AO to C, then

$\angle AOC = 180°$ and

$\angle BOC = 180° - 120° = 60°$

3. Let OD be the bisector of $\angle BOC$, then $\angle BOD = \frac{1}{2} \times 60° = 30°$,

so that $\angle AOD = \angle AOB + \angle BOD$

$= 120° + 30° = 150°$.

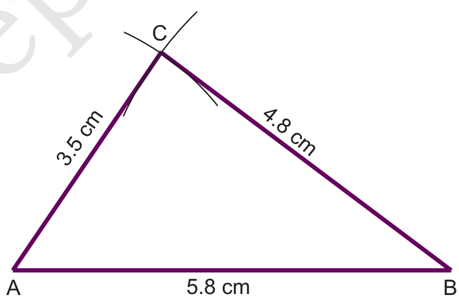
3.4. CONSTRUCTION OF TRIANGLES AND QUADRILATERALS

(a) Construction of Triangles

(i) To construct a triangle when the lengths of three sides are given

Example 1. *Construct a triangle ABC such that AB = 5.8 cm, BC = 4.8 cm and AC = 3.5 cm.*

Steps of construction



1. Draw line segment AB = 5.8 cm.

2. With B as centre and radius = BC = 4.8 cm, draw an arc,

3. With A as centre and radius = AC = 3.5 cm, draw an arc to cut the arc of step 2 at C.

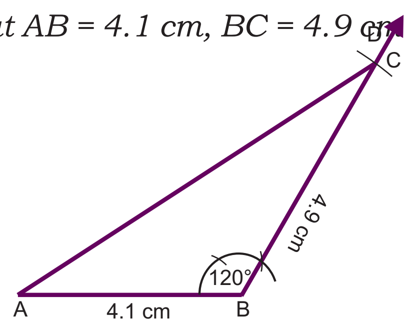
4. Join AC and BC.

Then ABC is the required triangle.

(ii) To construct a triangle when two sides and the included angle are given

Example 2. *Construct a triangle ABC such that AB = 4.1 cm, BC = 4.9 cm and ∠B = 120°.*

Steps of construction



1. Draw line segment AB = 4.1 cm.

2. At B, construct $\angle ABD = 120°$

3. From BD, cut off BC = 4.9 cm

4. Join AC.

Then ABC is the required triangle.

(iii) To construct a triangle when one side and two angles are given

When two angles say $\angle A$ and $\angle B$ of a triangle ABC are given, then since $\angle A + \angle B + \angle C = 180°$, the third angle $\angle C = 180° - \angle A - \angle B$ also becomes known.

Example 3. *Construct a triangle ABC such that BC = 7.5 cm, $\angle B = 45°$ and $\angle C = 60°$*

Steps of construction

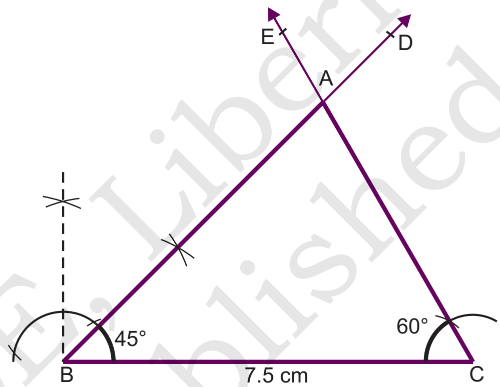
1. Draw line segment $BC = 7.5$ cm

2. At B, construct $\angle CBD = 45°$

3. At C, construct $\angle BCE = 60°$

4. Let BD and CE intersect at A.

Then ABC is the required triangle.



Example 4. *Construct an isosceles triangle ABC having base BC = 6.8 cm and $\angle B = 30°$.*

Since the base angles of an isosceles triangle are equal, therefore $\angle B = \angle C = 30°$

Steps of construction

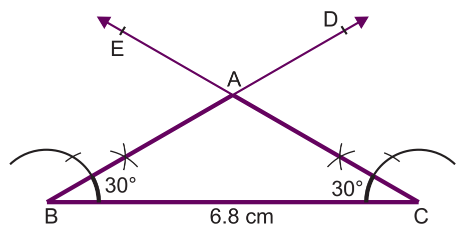
1. Draw line segment $BC = 6.8$ cm

2. At B, construct $\angle CBD = 30°$

3. At C, construct $\angle BCE = 30°$

4. Let BD and CE intersect at A.

Then ABC is the required isosceles triangle.



(iv) To construct a quadrilateral when three sides and two diagonals are given

Example 5. *Construct a quadrilateral ABCD in which AB = 5.5 cm, AD = 4.4 cm, CD = 6.5 cm , AC = 6.5 cm and BD = 7.1 cm.*

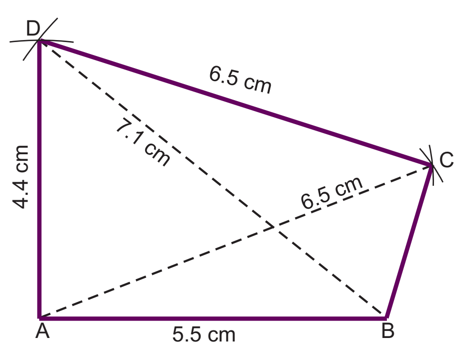
Steps of construction

1. Construct triangle ABD

2. Construct triangle ACD.

3. Join BC.

Then $ABCD$ is the required quadrilateral.



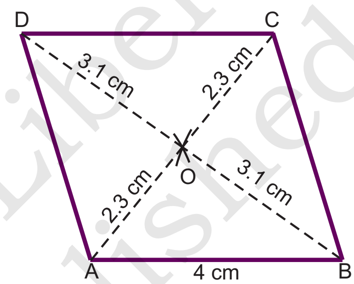
(v) To construct a parallelogram whose one side and both diagonals are given

To construct parallelograms, we use the following facts:

1. Opposite sides of a parallelogram are equal.

2. Diagonals bisect each other.

Example 6. *Construct a parallelogram ABCD given that AB = 4 cm, AC = 4.6 cm and BD = 6.2 cm.*



Steps of construction

1. Construct triangle OAB with

$AB = 4$ cm

$AO = \frac{1}{2}\ AC = \frac{1}{2} \times 4.6 = 2.3$ cm and

$BO = \frac{1}{2} BD = \frac{1}{2} \times 6.2 = 3.1$ cm.

2. Produce AO to C such that $OC = OA$

3. Produce BO to D such that $OD = OB$.

4. Join BC, CD, AD.

Then, $ABCD$ is the required parallelogram.

3.5. CONSTRUCTING LOCI

Locus is the set of all points in a plane which satisfy one or more geometrical conditions.

Alternatively, locus of a moving point is the path traced by the point under a given set of geometrical conditions.

Important facts about locus

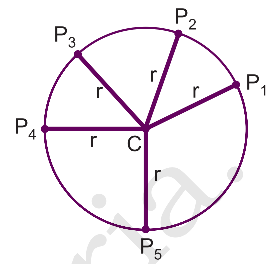
1. Every point which satisfies the given geometrical condition(*s*) lies on the locus.

2. Every point which lies on the locus satisfies the given geometrical condition(*s*).

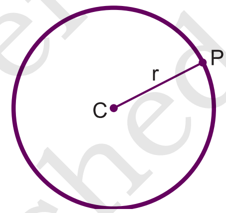
Thus, a point P lies on locus $\Leftrightarrow$ P satisfies the given geometrical condition(*s*).

Example 7. *Let $\{P_1, P_2, P_3, ...\}$ be the set of points in a plane such that their distance from a fixed point C in the plane is a positive constant r. Then, $CP_1 = CP_2 = CP_3 = ... = r$.*

Solution. We observe that the curve through all these point is a circle with the fixed point C as centre and the constant distance r as radius.



Conversely, if P is any point on the circle then its distance from the centre C is the radius r, i.e, $CP = r$ for all points P on the circle.



Thus, *a circle is the set of all points in a plane (i.e. locus) which are at a constant distance from a fixed point in the plane. The fixed point is the centre and the constant distance is the radius of the circle.*

EXERCISE

1. Construct a quadrilateral $ABCD$ such that $AB = 4.2$ cm, $BC = 3.7$ cm, $CD = 4.3$ cm, $AD = 3.1$ cm and $\angle A = 60°$.
2. Construct a quadrilateral $ABCD$ in which $AB = 4.4$ cm, $BC = 4$ cm, $CD = 6.4$ cm, $AD = 2.8$ cm and $BD = 6.6$ cm.
3. Draw a parallelogram $ABCD$ in which $AC = 6.8$ cm, $BD = 5$ cm and an angle between them is 60°.
4. Construct a triangle ABC in which $BC = 8$ cm, $AB = 5$ cm and $\angle ABC = 60°$. Mark a point P which is equidistant from AB, BC and also from B and C.
5. Draw a line segment of length 6.6 cm. Bisect it and measure the length of each part.
6. Construct a ΔABC in which $BC = 3.6$ cm, $AB + AC = 4.8$ cm and $\angle B = 60°$.

M12CH4

TOPIC

4

Statistics–I

4.1. BAR CHARTS

Bar charts are the most common type of diagrams used in practice. A bar is a rectangle whose width is insignificant. Only the length of the bar is significant. When the number of items is large, thick lines may be drawn instead of bars to economise space. For comparison, they are most effective.

A bar chart is a pictorial representation of the numerical data by a number of bars (rectangles) of uniform width drawn horizontally or vertically. The gap between one bar and another should be uniform throughout. The vertical bars are preferred because they give a better look and facilitate comparison. Each bar represents only one value of the numerical data and, therefore, there are as many bars as the number of values in the numerical data.

If the bars are drawn vertically on x-axis, then scale of heights of the bars is shown along y-axis. If the bars are drawn horizontally on y-axis, then scale of lengths of bars is shown along x-axis. The bars can be shaded or coloured.

The following examples will illustrate the construction of bar charts.

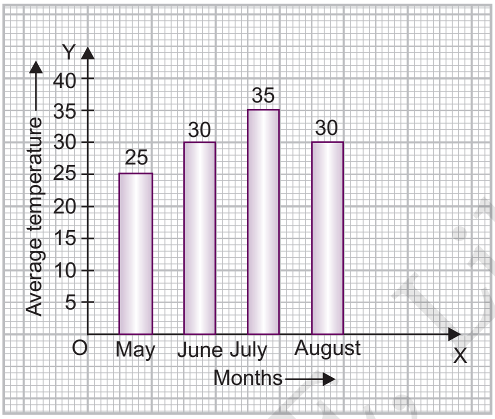
Example 1. *The average temperatures (in degrees celsius) for a city for the months May to August during a year as reported by the Meteorological Department are given as follows:*

Month	*May*	*June*	*July*	*August*
Average temperature	*25*	*30*	*35*	*30*

Represent the above data by bar chart.

Solution. Draw the mutually perpendicular lines OX and OY. Along OX, mark the months and along OY, the average temperature (in degrees celsius). Decide the width of each bar and the space to be left between consecutive bars. Choose a convenient scale on y-axis for the numerical data.

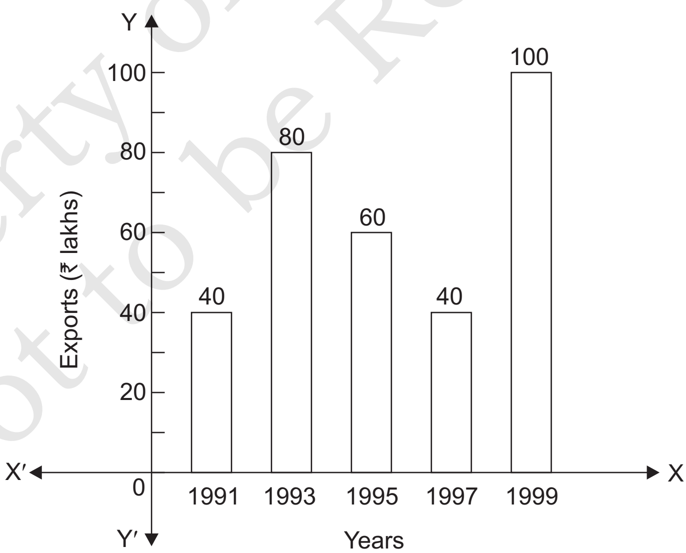
The bar chart for the given data is shown in figure.



Example 2. *Present the following information on exports of Liberia in the form of simple bar diagram:*

Years	*1991*	*1993*	*1995*	*1997*	*1999*
Exports (in L$)	*40*	*80*	*60*	*40*	*100*

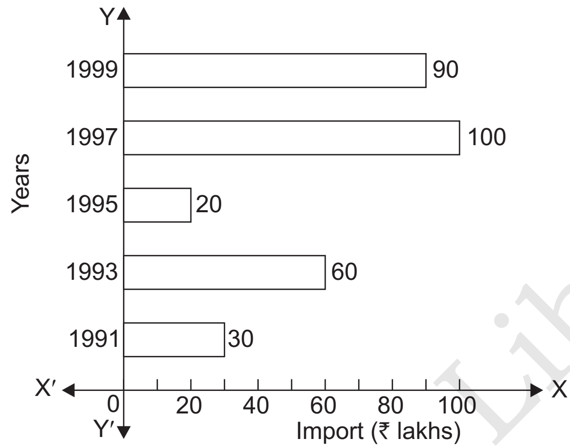
Solution.



Example 3. *Present the following information in a simple bar diagram (horizontal):*

Years	*1991*	*1993*	*1995*	*1997*	*1999*
Imports (in L$)	*30*	*60*	*20*	*100*	*80*

Solution.



4.2. PIE CHARTS

A pie chart is a pictorial representation of the numerical data by non-intersecting adjacent sectors of a circle such that area of each sector is proportional to the magnitude of the data represented by the sector. The pie chart is so called because the entire graph looks like a pie and the components (sectors) resemble the slices cut from a pie. Pie charts are used to show the break-up of a total into component parts. For example, the pie may represent the budget of a family for a month and the components may represent portions of the budget allotted to rent, food, clothing, education and so on. Similarly, through a pie chart it can be shown how an amount spent by a firm is distributed over various heads such as wages, raw materials, administration expenses etc.

Following steps may be used for the construction of a pie chart.

(*i*) The area of each sector is proportional to the component value of the component represented by the sector. Also, the area of a sector is proportional to the angle subtended by its arc at the centre, therefore, central angles are proportional to the component values.

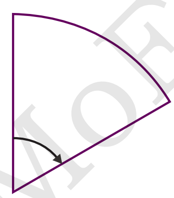
Thus, central angle of a component

$$= \left(\frac{\text{Value of the component}}{\text{Sum of the component values}} \times 360 \right)^{\circ}$$

Using the above formula, find the central angles of the various components and arrange them in descending order.

(*ii*) Draw a circle of appropriate size with a compass. Also, draw the vertical radius coinciding with 12 o'clock position of the two hands of a clock.

(*iii*) In laying out the sectors of a pie chart figure, it is desirable to follow some logical arrangement. It is common procedure to arrange the sectors according to size, with the largest at the top and the others in succession running clockwise. Thus start with the component having largest central angle and construct a sector in such a way that its one radius coincides with the vertical radius drawn in step (*ii*) and the other radius is below the first radius in clockwise direction.



(*iv*) Construct other sectors representing other component in clockwise sense in descending order of magnitude of central angles. Components like 'Miscellaneous', irrespective of the magnitude of their central angle, are shown last.

(*v*) Shade the sectors by different colours or designs.

(*vi*) An essential feature of a pie chart is to put descriptive labels inside each sector so that they can be easily identified. If it is not possible to place the labels inside the sectors due to lack of space, then put the lables outside the circle with an arrow pointing to the appropriate sector.

Example 4. *Draw a pie chart for the following data of expenditure pattern in a family:*

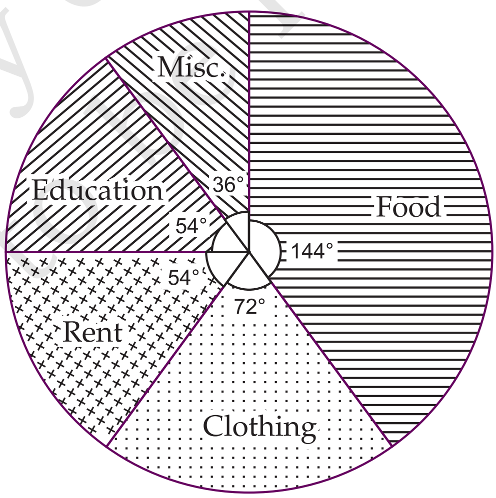
Items	*Food*	*Clothing*	*Rent*	*Education*	*Miscellaneous*
Expenditure (in cedis)	*4000*	*2000*	*1500*	*1500*	*1000*

Solution.

Computation of Central Angles

Item	Expenditure	Central angle
Food	4000	$\left(\frac{4000}{10000} \times 360\right)^\circ = 144^\circ$
Clothing	2000	$\left(\frac{2000}{10000} \times 360\right)^\circ = 72^\circ$
Rent	1500	$\left(\frac{1500}{10000} \times 360\right)^\circ = 54^\circ$
Education	1500	$\left(\frac{1500}{10000} \times 360\right)^\circ = 54^\circ$
Miscellaneous	1000	$\left(\frac{1000}{10000} \times 360\right)^\circ = 54^\circ$
Total	10,000	360°

Draw a circle of appropriate size with a compass. Draw the vertical radius. Construct sectors in clockwise sense with descending order of magnitude of central angles as shown in figure.



Example 5. *The number of students admitted in different faculties of a college are given below:*

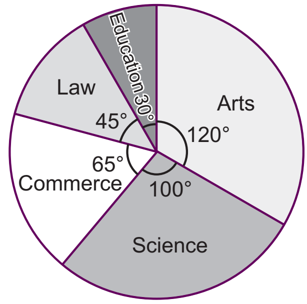
Faculty	*Number of students*
Science	*1000*
Arts	*1200*
Commerce	*650*
Law	*450*
Education	*300*
Total	*3600*

Draw a pie chart to represent the above information.

Solution. Computation of Central Angles

Faculty	**Number of students**	**Central angle**
Science	1000	$\left(\frac{1000}{3600}\times 360\right)^\circ = 100^\circ$
Arts	1200	$\left(\frac{1200}{3600}\times 360\right)^\circ = 120^\circ$
Commerce	650	$\left(\frac{650}{3600}\times 360\right)^\circ = 65^\circ$
Law	450	$\left(\frac{450}{3600}\times 360\right)^\circ = 45^\circ$
Education	300	$\left(\frac{300}{3600}\times 360\right)^\circ = 30^\circ$
Total	3600	360°

Draw a circle of appropriate size with a compass. Draw the vertical radius. Construct sectors in clockwise sense with descending order of magnitude of central angles as shown in figure.



4.3. HISTOGRAMS

A histogram is a graphical representation of a frequency distribution in the form of adjacent rectangles with class intervals as bases and heights proportional to frequency.

Note that in a bar chart, there is uniform gap between one bar and another whereas there is no gap between adjacent rectangles in a histogram. In a bar chart, the width of a rectangle is insignificant. Whereas in a histogram, the width of a rectangle is significant and represents the class size.

To construct a histogram, we proceed as follow:

(*i*) Represent the class intervals along x-axis on a suitable scale.

(*ii*) Represent the frequency along y-axis on a suitable scale.

(*iii*) Construct rectangles with class intervals as bases and the corresponding frequencies as heights.

The diagram representing the above rectangles is the required histogram.

Note: It must not be assumed that scale for both the axes is the same. We can have different scales for the two axes. The determination of scale depends upon our convenience and the type of data. The scale or scales should be so chosen as to fit the size of graph paper and to hold all the figures of the data.

4.4. MEAN OF GROUPED DATA WITH CONTINOUS FREQUENCY DISTRIBUTION

In a grouped data with continuous or exclusive classes, we may recall that an observation falling in any upper-class limit it considered in the next class. Here we assume that the frequency of each class interval is centred around its midpoint, called the class mark. So the mark of each class-interval serves a representative of the whole class. The class mark of any class is equal to the average of its upper and lower limits. Thus,

$$\text{Class mark} = \frac{\text{Upper class limit} + \text{Lower class limit}}{2}$$

4.5. FREQUENCY TABLE

As soon as the work related to collection of data is over, the investigator has to find out ways to present them in a form which is meaningful, easily understood and gives the main feature at a glance. Usually, data available to us is in unorganised form called *raw data.*

Making Frequency Tables

In this section, we will learn how to make frequency tables by tallying in groups of five and write the frequencies. Let us construct frequency tables for a given data.

Example 6. *A group of 30 pupils were surveyed on which animal they would like the most to have as a pet. The results are given below:*

dog, cat, cat, fish, cat, rabbit, dog, cat, rabbit, dog, cat, dog, dog, dog, cat, rabbit, fish, rabbit, dog, cat, dog, cat, cat, dog, rabbit, cat, fish, dog, rabbit, cat.

Make a frequency distribution table for this data.

Solution. For counting purposes, we use tally marks. After putting 4 tally marks vertically, we cross it as shown below and again we take the tally marks in the same manner, counting in sets of fives.

Frequency Distribution Table

Animal	***Tally Marks***	***Number of Animals (Frequency)***
Dog	𝍸 𝍸	10
Cat	𝍸 𝍸 |	11
Fish	|||	3
Rabbit	𝍸 |	6
	Total	30

As mentioned earlier, the frequency gives the number of times a particular entry occurs. The above table is known as *frequency distribution table* as it gives the number of times an entry occurs.

Grouping Data

Sometimes, we have to deal with a large data.

If we make a frequency distribution table for each observation, the table would be too long, so, for convenience, we make groups of observations say, 0–10, 10–20 and so on, and obtain a frequency distribution of the number of observations falling in each group.

Example 7. *Consider the following marks (out of 50) obtained in Mathematics by 60 pupils of Junior High School 2.*

21, 10, 30, 22, 33, 5, 37, 12, 25, 42, 15, 39,
26, 32, 18, 27, 28, 19, 29, 35, 31, 24, 36, 18,
20, 38, 22, 44, 16, 24, 10, 27, 39, 28, 49, 29,
32, 23, 31, 21, 34, 22, 23, 36, 24, 36, 33, 47,
48, 50, 39, 20, 7, 16, 36, 45, 47, 30, 22, 17.

Make a frequency table for the above data using intervals 0–10, 10–20 and so on.

Solution. Grouped Frequency Distribution Table

Groups	***Tally Marks***	***Frequency***
0–10	||	2
10–20	𝍸 𝍸	10
20–30	𝍸 𝍸 𝍸 𝍸 |	21
30–40	𝍸 𝍸 𝍸 ||||	19
40–50	𝍸 ||	7
50–60	||	1
	Total	**60**

Data represented in this way is said to be *grouped* and the distribution obtained is called *grouped frequency distribution table.*

It helps us to draw meaningful inferences like:

(*i*) Most of the pupils have scored between 20 and 40.

(*ii*) Eight pupils have scored more than 40 marks out of 50 and so on.

Each of the groups: 0–10, 10–20, 20–30 etc., is called a *Class Interval* (or briefly a class).

Observe that 10 occurs in both classes, *i.e.*, 0–10 as well as 10–20. Similarly, 20 occurs in classes 10–20 and 20–30. But it is not possible that an observation (say 10 or 20) can belong simultaneously to two classes. To avoid this, we adopt the convention that the common observation will belong to the higher class, *i.e.,* 10 belongs to the class interval 10–20 (and not to 0–10).

Similarly, 20 belongs to 20–30 (and not to 10–20). In the class interval, 10–20, 10 is called the *lower class limit* and 20 is called the *upper class limit.*

Similarly, in the class interval 20–30, 20 is the *lower class limit* and 30 is the *upper class limit.*

Observe that the difference between the upper class limit and lower class limit for each of the class intervals 0–10, 10–20, 20–30 etc., is equal, (10 in this case). This difference between the upper class limit and lower class limit is called the *width* or *size* of the class interval.

4.6. MEAN OF GROUPED DATA WITH CONTINUOUS FREQUENCY DISTRIBUTION

In a grouped data with continuous or exclusive classes, we may recall that an observation falling in any upper-class limit is considered in the next class. Here we assume that the frequency of each class interval is centered arounts its midpoint, called the class mark, So the class mark of each class interval serves a representative of the whole class. The class mark of any class is equal to the average of its upper and lower limits. Thus

$$\text{Class mark} = \frac{\text{Upper class limit} + \text{Lower class limit}}{2}$$

Mean of Grouped data by direct method

Step 1 For each class, find the class mark x_i

$$x_i = \frac{1}{2}(\text{lower limit} + \text{upper limit})$$

Step 2 Calculate $f_i x_i$ for each class

Step 3 Calculate the mean $\bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i}$

This method is used when the numerical values of f_i and x_i are small.

Example 8. *A survey was conduction by a group of students as a part of their environment awareness programme, in which they collected the following data regarding the number of plants in 20 houses in a locality. Find the mean number of plants per house.*

Number of plants	0–2	2–4	4–6	6–8	8–10	10–12	12–14
Number of houses	1	2	1	5	6	2	3

Which method did you use for finding the mean, and why?

Solution. Grouped Frequency Distribution Table

Number of plants (C.I.)	**Number of Houses (f_i)**	**Class mark (x_i)**	$f_i x_i$
0–2	1	1	1
2–4	2	3	6
4–6	1	5	5
6–8	5	7	35
8–10	6	9	54
10–12	2	11	22
12–14	3	13	39
Total	$\Sigma f_i = 20$		$\Sigma f_i x_i = 162$

$$\therefore \quad \text{Mean } (\bar{x}) = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{162}{20} = 8.1$$

Hence, the mean number of plants per house is 8.1. Here, we have used direct method for finding the mean because the values of f_i and x_i are small.

4.7. MODE

Mode of any frequency distribution is that value of the observation for which the frequency is maximum. A given frequency distribution may have the some maximum frequency for more than one observation then given data is multi modal.

Mode of Grouped Data:

(*i*) From the given frequency distribution, determine the class of maximum frequency distribution. This is known as modal class.

(*ii*) Calculate the mode by using the formula

$$\text{Mode} = l + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times h$$

where l = lower level of modal class

f_i = frequency of modal class

f_0 = frequency of the class preceding the modal class

f_2 = frequency of the class succeeding the modal class

h = class size

Example 9. *Find the mode of the following data:*

Marks	50–60	60–70	70–80	80–90	90–100
Number of students	3	12	32	20	6

Solution. As the class interbal 70–80 has maximum frequency i.e., 32 the modal class 70–80.

Now $l = 70$, $f_1 = 32$, $f_0 = 12$, $f_2 = 20$, $h = 10$

$$\therefore \quad \text{Mode} = l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right)h = 70\left(\frac{32 - 12}{2 \times 32 - 12 - 20}\right) \times 10$$

$$= 70 + \frac{20}{32} \times 10 = 70 + 6.25 = 76.25$$

4.8. MEDIAN OF A GROUPED FREQUENCY DISTRIBUTION

1. Prepare a cumulative frequency table from the given frequency distribution
2. Find n(total frequency) and calculate $\frac{n}{2}$
3. From the cumulative frequency column of the table, find the cumulative frequency.
4. Calculate the median, by using the formula

$$\text{Median} = l + \left(\frac{\frac{n}{2} - Cf}{f}\right) \times h$$

where l = lower limit of median class

Cf = Cumulative frequency of the class preceding the median class

f = Frequency of median class

h = Class size

Example 10. *Weekly income of 600 families in tabulated below:*

Weekly income (in L$)	*Number of families*
0–1000	*250*
1000–2000	*190*
2000–3000	*100*
3000–4000	*40*
4000–5000	*15*
5000–6000	*5*
Total	*600*

Compute the median income.

Solution. Frequency distribution table is

Weekly income (in L$)	*Number of families*	*Cumulative frequency*
0–1000	250	250
1000–2000	190	440 ← Median class
2000–3000	100	540
3000–4000	40	580
4000–5000	15	595
5000–6000	5	600

Here $n = 600$

$\therefore$ $\frac{n}{2} = 300$

$\therefore$ Median class is 1000–2000

Now $l = 1000$, $Cf = 250$, $f = 190$, $h = 100$

$$\text{Median} = l + \left(\frac{\frac{n}{2} - Cf}{f}\right) \times h = 1000 + \left(\frac{300 - 250}{190}\right) \times 1000$$

$$= 1000 + 263.16 = 1263.16$$

4.9. QUARTILES AND PERCENTILES

Quartiles: Distribution is divided into quarters.

Percentiles: Distribution is divided into hundredths.

4.10. CUMULATIVE FREQUENCY CURVE (OGIVE)

The curve obtained by plotting cumulative frequencies is called a **cumulative frequency curve** or an **ogive**.

Plot the points with the *upper limits* of the classes as abscissae and the corresponding less than cumulative frequencies as ordinates. Join the points by a free hand smooth curve to get the required ogive. It is a rising curve.

Example 11. *Draw the ogive for the following distribution showing the number of marks of 59 pupils:*

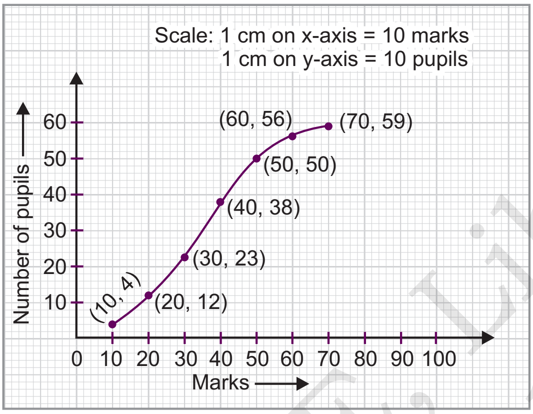
Marks	*No. of students*	*Marks*	*No. of students*
0–10	4	40–50	12
10–20	8	50–60	6
20–30	11	60–70	3
30–40	15	—	—

Solution. We first calculate the cumulative frequencies as shown in the following table.

Table. Table of cumulative frequencies

Marks	*Number of pupils*	*Upper limits*	*Cumulative frequency (less than)*
0–10	4	10	4
10–20	8	20	12
20–30	11	30	23
30–40	15	40	38
40–50	12	50	50
50–60	6	60	56
60–70	3	70	59

Plot the points (10, 4), (20, 12), (30, 23), (40, 38), (50, 50), (60, 56) and (70, 59) on the graph paper and join them by free hand. The smooth rising curve so obtained is the required ogive. (see figure).



Ogive or cumulative frequency curve

Through Q, draw a straight line parallel to *y*-axis to meet the *x*-axis at the point Q_1 (say). The abscissa of Q_1 is L$ 1725.

∴ D_9 = L$ 1725 is the required decile

Calculation of P_5, P_{30} and P_{83}:

For P_5; $\frac{5N}{100} = \frac{5\times100}{100} = 5$

For P_{30}; $\frac{30N}{100} = \frac{30\times100}{100} = 30$

For P_{83}; $\frac{83N}{100} = \frac{83\times100}{100} = 83$

Draw a straight line through the points 5, 30 and 83 (on *y*-axis) parallel to *x*-axis to meet the ogive at the points R, S and T (say).

Through R, S and T, draw straight lines parallel to *y*-axis to meet the x-axis at the points R_1, S_1 and T_1.

The abscissa of R_1 is L$ 366.

∴ P_5 = L$ 366

The abscissa of S_1 is L$ 857.

∴ P_{30} = L$ 857

The abscissa of T_1 is L$ 1560.

∴ P_{83} = L$ 1560

Hence, P_5 = L$ 366, P_{30} = L$ 857 and
P_{83} = L$ 1560 are the required percentiles.

4.11. QUARTILES DEFINITION

Quartlies divide the entire set into four equal parts. So, there are three quartiles, first, second and third represented by Q_1, Q_2 and Q_3. Q_2 is nothing but the median, since it indicate the position of the item in the list and thus, is a positional average. To find quartiles of a group of data, we have to arrange the data in ascending order.

4.12. PERCENTILE

A percentiles (or a percentile) is a measure used in statistics indicating the value *below which* a given percentage of observations in a group of observations fall. For example, the 20th percentile is the value (or score) below which 20% of the observation may be found.

The term percentile and the related term *percentile rank* are often used in the reporting of scores from norm-references tests. For example, if a score is at the 86th percentile, where 86 is the percentile rank, it is equal to the value below which 86% of the observations may be found. In contrast, iif it is in the 86th percentile, the score is at or below the value of which 86% of the observations may be found. ***Every score is in the 100th percentile.***

The 25th percentile is also known as the first quartile (Q1), the 50th percentile as the median or second quartile (Q2) and the 75th percentile as the third quartile (Q3). In general, percentiles and quartiles are specific types of quantiles.

Formula: $Ly = \dfrac{(n+1)Y}{100}$

Example 12. *Calculate the first quartile of a distribution that consists of the following portfolio returns.*

3% 4% 6% 9% 11% 12% 14%

Solution. The first quartile = (7 + 1)25% = 2^{nd} item in the data set (4%) i.e. 25% of the observation lie below the second observation from the left.

EXERCISE

1. The import and export of a country (in million dollars) during the years (2012–13 to 2016–17) are given below:

Year:	2012–13	2013–14	2014–15	2015–16	2016–17
Import:	4000	5000	7000	8000	9500
Export:	5000	7000	8500	11000	14000

2. Present the following information on expenditure by family A and family B in a multiple bar diagram:

Items of expenditure	*Family A*	*Family B*
Food	10000	7000
Clothing	3000	1000
Education	5000	1500
Misc. exp.	2000	500

3. Draw a pie diagram to represent the following information of expenditure by a family:

Items of expenditure	Food	Education	Housing	Clothing	Misc.
% of total expenditure	60	15	10	10	5

4. A Junior Football Club in Accra has thirty members. Their ages are given below. Make a frequency distribution table for it.

13, 17, 13, 13, 14, 15, 15, 14, 16, 14, 16, 17, 16, 16, 13, 15, 16, 15, 15, 14, 14, 15, 15, 13, 13, 15, 14, 13, 16, 17.

5. The monthly wages (in L$) of 30 workers in a factory are:

330, 335, 390, 310, 335, 336, 369, 345, 398, 390, 360, 332, 333, 355, 345, 304, 308, 312, 340, 385, 335, 335, 336, 378, 340, 368, 390, 306, 340, 320.

Using tally marks make a frequency table with intervals as 300–310, 310–320 and so on.

6. Following is the distribution of marks in Mathematics test at Senior High School 2 level

Marks	*Number of Students*
0–10	3
10–20	7
20–30	10
30–40	20
40–50	6
50–60	4

If 60% of the students pass the test, use cumulative frequency curve to find the minimum marks obtained by a pass student.

7. Find the median, lower quartile and upper quartile of the following numbers:

12, 5, 22, 30, 7, 36, 14, 42, 15, 53, 25

8. The distribution of heights of 50 students (in nearest cm) is given below:

Height (in cm)	110	115	118	120	121	125
Number of students	6	8	14	15	4	3

9. Find the mean of the following frequency distribution:

Class	0–100	100–200	200–300	300–400	400–500
Frequency	6	9	15	12	8

PERIOD–II

M12CH5

TOPIC 5

Standard Deviation

5.1. MEANING

Suppose there are two classes of students with average percentage of marks 56% each, it would not be justified to take for guaranteed that the students of both classes are approximately of same level of intelligence. It may be just possible that the students of first class are getting marks between 50% and 59% and the students of the second class getting between 8% and 90%. Thus, we see that even if the average value of two series are same, the items may be of quite different in nature, in two series. The measures of central tendency are inadequate to give the complete idea about the items in the series, we would be very much helped, if we are also given some idea about the scatter of items about the average value. The measure of scatteredness of items about some average is also called measure of dispersion. We may define measure of dispersion for a statistical data as the measurement of the spread of the items about an average value.

5.2. METHODS OF MEASURING DISPERSION

I. Range
II. Quartile Deviation (Q.D.)
III. Mean Deviation (M.D.)
IV. Standard Deviation (S.D.)
V. Lorenz Curve.

I. RANGE

DEFINITION

The **range** of a statistical data is defined as the difference between the largest and the smallest values of the variable.

∴ **Range = L – S**,

where L = largest value of the variable

S = smallest value of the variable.

In case, the values of the variable are given in the form of classes, then L is taken as the upper limit of the largest value class and S as the lower limit of the smallest value class.

Example 1. *Find the range of the series :*

4, 2, 6, 8, 10.

Solution. Here L = 10, S = 2.

∴ Range = L – S = 10 – 2 = **8.**

Example 2. *Find the range of the following distribution* :

No. of students	*16–18*	*18–20*	*20–22*	*22–24*	*24–26*	*26–28*
No. of students	*0*	*4*	*6*	*8*	*2*	*2*

Solution. Here L = 28, S = 18

∴ Range = L – S = 28 – 18 = **10 years.**

It may be noted that S ≠ 16, though it is the lower limit of the smallest value class, but there is no item in this class and so this class is meaningless so far as the calculation of range is concerned.

Let us consider the market value of shares of companies A and B, during a particular week.

Day	***Monday***	***Tuesday***	***Wednesday***	***Thursday***	***Friday***	***Saturday***
M.V. of shares of company A (in L$)	12	11	10	13	16	20
M.V. of shares of company B (in L$)	60	50	55	62	70	75

From the data, we see that Range (A) = 20 – 10 = L$ 10 and Range (B) = 75 – 50 = L$ 25. From these results, one is likely to infer that there is more variability in the II series. But this is not so, because the M.V. of shares of A has increased by 100% in the week, whereas there is only 50% rise in the M.V. of shares of B, during that week. Thus, variability is more in the first series. Thus, we see that range may give misleading results if used for comparing two or more series for variability (scatteredness, dispersion). For comparison purpose, we use its corresponding relative measure, called '*coefficient of range*'. This is defined as

$$\textbf{Coeff. of Range} = \frac{L - S}{L + S}.$$

Now Coeff. of Range for A $= \frac{20 - 10}{20 + 10} = \frac{10}{30} = 0.3333.$

Coeff. of Range for B $= \frac{75 - 50}{75 + 50} = \frac{25}{125} = 0.2000.$

$\therefore$ Coeff. of Range (A) > Coeff. of Range (B)

$\therefore$ Variability is more in the M.V. of shares of company A.

II. QUADRTILE DEVIATION (Q.D.)

5.3. INADEQUACY OF RANGE

Consider the series

I : 4, 4, 4, 5, 5, 6, 4, 5, 5, 1000.

II : 4, 4, 4, 5, 5, 6, 4, 5, 5.

For series I, Coeff. of Range $= \frac{1000 - 4}{1000 + 4} = \frac{996}{1004} = 0.992$

For series II, Coeff. of Range $= \frac{6 - 4}{6 + 4} = \frac{2}{10} = 0.200.$

On comparing the values of coeff. of range for these series, one is likely to conclude that these is marked difference in variability in the series. In fact, the series II is obtained from the series I, just by ignoring the extreme item 1000. Thus, we see that extreme items can distort the value of range and even the coefficient of range. If we have a glance at the definitions of these measures, we would find that only extreme items are required in their calculation, if at all extreme items are present. Even if extreme items are present in a series, the middle 50% values of the variable would be expected to vary quite smoothly, keeping this in view, we define another measure of dispersion, called 'Quartile Deviation'.

5.4. QUARTILE DEVIATION

The **quartile deviation** of a statistical data is defined as

$\frac{Q_3 - Q_1}{3}$ and is denoted as Q.D.

This is also called *semi-inter quartile* range. We have already studied the method of calculating quartiles. The value of Q.D. is obtained by subtracting Q_1 from Q_3 and then dividing it by 2.

For comparing two or more series for variability, the absolute measure Q.D. would not work. For this purpose, the corresponding relative measure, called coeff. of Q.D. is calculated. This is defined as :

$$\textbf{Coeff. of Q.D.} = \frac{\mathbf{Q_3 - Q_1}}{\mathbf{Q_3 + Q_1}}.$$

Example 3. *Find Q.D. and its coefficient for the following series:*

x (in L$) : 4, 7, 6, 5, 9, 12, 19.

Sol. The values of the variable arranged in ascending order are

x (in L$) : 4, 5, 6, 7, 9, 12, 19.

Here $n = 7$.

$\mathbf{Q_1}$: $\frac{n+1}{4} = \frac{7+1}{4} = 2$ ∴ Q_1 = size of 2nd item = in L$ 5

$\mathbf{Q_3}$: $3\left(\frac{n+1}{4}\right) = 3\left(\frac{7+1}{4}\right) 6$ ∴ Q_3 = size of 6th item = in L$ 12

$$\therefore \quad \text{Q.D.} = \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{12-5}{2} = \textbf{in L\$ 3.5.}$$

$$\text{Coeff. of Q.D.} = \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{12-5}{12+5} = \frac{7}{17} \ \mathbf{0.4118.}$$

Example 4. *Find the quartile deviation for the following distribution:*

Marks	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
No. of students	*10*	*11*	*12*	*13*	*5*	*12*	*7*	*5*

Sol. **Calculation of Quartiles**

Marks	*No. of students* *f*	*c.f.*
2	10	10
3	11	21
4	12	33
5	13	46
6	5	51
7	12	63
8	7	70
9	5	75 = N
	N = 75	

$$Q_1 : \frac{N+1}{4} = \frac{75+1}{4} = 19$$

$\therefore$ Q_1 = size of 19th item = 3 marks

$$Q_3 : 3\left(\frac{N+1}{4}\right) = 3\left(\frac{75+1}{4}\right) = 57$$

$\therefore$ Q_3 = size of 57th item = 7 marks

$$\therefore \quad \text{Q.D.} = \frac{Q_3 - Q_2}{2} = \frac{7-3}{2} = \textbf{2 marks.}$$

III. MEAN DEVIATION (M.D.)

5.5. MEAN DEVIATION (M.D.)

Mean deviation is also called **average deviation**. The **mean deviation** of a statistical data is defined as the arithmetic mean of the numerical values of the deviations of items from some average. Generally, A.M. and median are used in calculating mean deviation. Let '*a*' stand for the average used for calculating M.D.

For an **individual series**, the M.D. is given by

$$\textbf{M.D.} = \frac{\sum_{i=1}^{n} |x_i - a|}{n} = \frac{\Sigma |x - a|}{n}$$

where $x_1, x_2, \ldots\ldots, x_n$ are the values of the variable, under consideration.

For a **frequency distribution,**

$$\textbf{M.D.} = \frac{\sum_{i=1}^{n} f_i \,|\, x_i - a \,|}{N} = \frac{\Sigma f \,|\, x - a \,|}{N}$$

where f_i is the frequency of x_i $(1 \le i \le n)$.

When the values of the variable are given in the form of classes, then their respective mid-points are taken as the values of the variable.

Median is used in calculating M.D., because of its property that the sum of numerical values of deviations of items from median is always least. So, if median is used in the calculation of M.D., its value would come out to be least. M.D. is also calculated by using A.M. because of its simplicity and popularity. In problems, it is generally given as to which average is to be used in the calculation of M.D. If it is not given, then either of the two can be made use of.

5.6. COEFFICIENT OF M.D.

For comparing two or more series for variability, the corresponding relative measure, 'Coefficient of M.D.', is used. This is defined as :

$$\textbf{Coeff. of M.D.} = \frac{\textbf{M.D.}}{\textbf{Average}}.$$

If M.D. is calculated about A.M., then M.D. is written as M.D.. Similarly, M.D.(Median) would mean that median has been used in calculating M.D.

∴ We can write

$$\textbf{Coeff. of M.D.}\ (\bar{x}) = \frac{\textbf{M.D.}(\bar{x})}{\bar{x}}$$

$$\textbf{Coeff. of M.D.(Median)} = \frac{\textbf{M.D.(Median)}}{\textbf{Median}}$$

WORKING RULES TO FIND M.D. ($\bar{x}$)

Rule I. *In case of an individual series, first find $\bar{x}$ by using the formula $\bar{x} = \frac{\Sigma x}{n}$. In the second step, find the values of $x - \bar{x}$. In the* *next step, find the numerical values* $|x - \overline{x}|$ *of* $x - \overline{x}$. *Find the sum* $\Sigma|x - \overline{x}|$ *of these numerical values* $|x - \overline{x}|$. *Divide this sum by n to get the value of M.D.*$(\overline{x})$.

Rule II. *In case of a frequency distribution, first find* $\overline{x}$ *by using the formula* $\overline{x} = \frac{\Sigma fx}{N}$. *In the second step, find the values of* $x - \overline{x}$. *In the next step, find the numerical values* $|x - \overline{x}|$ *of* $x - \overline{x}$. *Find the products of the values of* $|x - \overline{x}|$ *and their corresponding frequencies. Find the sum* $\Sigma f|x - \overline{x}|$ *of these products. Divide this sum by N to get the value of M.D.*$(\overline{x})$.

Rule III. *If the values of the variable are given in the form of classes, then their respective mid-points are taken as the values of the variable.*

Rule IV. *To find the coefficient of M.D.*$(\overline{x})$, *divide M.D.*$(\overline{x})$ *by* $(\overline{x})$.

Example 5. *Find M.D. and M.D.(median) for the following statistical series :*

7, 10, 12, 13, 15, 20, 21, 27, 30, 35.

Solution. **Calculation of M.D.**

S. No.	x	$x - \overline{x}$ $\overline{x} = 19$	$\lvert x - \overline{x} \rvert$
1	7	– 12	12
2	10	– 9	9
3	12	– 7	7
4	13	– 6	6
5	15	– 4	4
6	20	1	1
7	21	2	2
8	27	8	8
9	30	11	11
10	35	16	16
$n = 10$	$\Sigma x = 190$		$\Sigma \lvert x - \overline{x} \rvert = 76$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{190}{10} = 19$$

$$\therefore \quad \text{M.D.}(\bar{x}) = \frac{\Sigma|x - \bar{x}|}{n} = \frac{76}{10} = 7.6.$$

Calculation of M.D.(median)

S. No.	***x***	***x – median*** ***median = 17.5***	***| x – median |***
1	7	– 10.5	10.5
2	10	– 7.5	7.5
3	12	– 5.5	5.5
4	13	– 4.5	4.5
5	15	– 2.5	2.5
6	20	2.5	2.5
7	21	3.5	3.5
8	27	9.5	9.5
9	30	12.5	12.5
10	35	17.5	17.5
$n = 10$			$\Sigma\| x - \text{median} \| = 76$

$$\frac{n+1}{2} = \frac{10+1}{2} = \mathbf{5.5}$$

$$\therefore \quad \text{Median} = \frac{\text{Size of 5th item + size of 6th item}}{2} = \frac{15+20}{20}$$

$$= 17.5$$

$$\therefore \quad \text{M.D.(median)} = \frac{\Sigma|x - \text{median}|}{n} = \frac{76}{10} = \mathbf{7.6.}$$

Example 6. *Find the coeff. of M.D.(Median) for the following frequency distribution:*

Marks	*0–10*	*10–20*	*20–30*	*30–40*	*40–50*
No. of students	*5*	*8*	*15*	*16*	*6*

Solution. **Calculation of M.D.(Median)**

Marks	*No. of students*	*c.f.*	*Mid-points of classes x*	*x-median (med. = 28)*	*| x – med. |*	*f| x – med. |*
0–10	5	5	5	– 23	23	115
10–20	8	13	15	– 13	13	104
20–30	15	28	25	– 3	3	45
30–40	16	44	35	7	7	112
40–50	6	50 = N	45	17	17	102
	N = 50					$\Sigma f \mid x - med. \mid = 478$

Median = size of 50/2th item = size of 25th item.

∴ Median class is 20–30

$$\text{Median} = L + \left(\frac{N/2 - c}{f}\right)h = 20 + \left(\frac{25 - 13}{15}\right)10$$

$$= 28$$

Now $$\text{M.D.(Median)} = \frac{\Sigma f \mid x - \text{median} \mid}{N} = \frac{478}{50} = 9.56 \text{ marks.}$$

$$\therefore \text{ Coeff. of M.D.(Median)} = \frac{\text{M.D.(Median)}}{\text{Median}} = \frac{9.56}{28} = \mathbf{0.3414.}$$

IV. STANDARD DEVIATION (S.D.)

5.7. STANDARD DEVIATION (S.D.)

It is the most important measure of dispersion. It finds indispensable place in advanced statistical methods. The **standard deviation** of a statistical data is defined as the positive square root of the A.M. of the squared deviations of items from the A.M. of the series under consideration. The S.D. is often denoted by the greek letter 'σ'.

For an **individual series**, the S.D. is given by

$$\textbf{S.D.} = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}} = \sqrt{\frac{\Sigma (x - \bar{x})^2}{n}}$$

where $x_1, x_2, \ldots\ldots, x_n$ are the value of the variable, under consideration.

For a **frequency distribution**,

$$\textbf{S.D.} = \sqrt{\frac{\sum_{i=1}^{n} f_i (x_i - \bar{x})^2}{N}} = \sqrt{\frac{\Sigma f(x - \bar{x})^2}{N}}$$

where f_i is the frequency of x_i $(1 \le i \le n)$.

When the values of the variable are given in the form of classes, then their respective mid-points are taken as the values of the variable.

WORKING RULES TO FIND S.D.

Rule I. *In case of an individual series, first find* $\bar{x}$ *by using the formula* $\bar{x} = \frac{\Sigma x}{n}$. *In the second step, find the values of* $x - \bar{x}$. *In the next step, find the numerical values* $(x - \bar{x})^2$ *the values of* $x - \bar{x}$. *Find the sum* $\Sigma(x - \bar{x})^2$ *of these numerical values* $(x - \bar{x})^2$. *Divide this sum by n. Take the positive square root of this to get the value of S.D.*

Rule II. *In case of a frequency distribution, first find* $\bar{x}$ *by using the formula* $\bar{x} = \frac{\Sigma x}{n}$. *In the second step, find the values of* $x - \bar{x}$. *In the next step, find the squares* $(x - \bar{x})^2$ *of the values of* $x - \bar{x}$. *Find the products of the values of* $(x - \bar{x})^2$ *and their corresponding frequencies. Find the sum* $\Sigma f(x - \bar{x})^2$ *of these products. Divide this sum by N. Take the positive square root of this to get the value of S.D.*

Rule III. *If the values of the variable are given in the form of classes, then their respective mid-points are taken as the values of the variable.*

Rule IV. (*i*) *Coeff. of S.D.* $= \frac{S.D.}{A.M.}$

Example 7. *Find the S.D. for the following data:*

4, 6, 10, 12, 18.

Sol. **Calculation of S.D.**

S. No.	*x*	$x - \bar{x}$ $\bar{x} = 10$	$(x - \bar{x})^2$
1	4	– 6	36
2	6	– 4	16
3	10	0	0
4	12	2	4
5	18	8	64
$n = 5$	$\Sigma x = 50$		$\Sigma(x - \bar{x}) = 120$

$$\bar{x} = \frac{\Sigma x}{n} = \frac{50}{5} = 10$$

Now $$\text{S.D.} = \sqrt{\frac{\Sigma (x - \bar{x})^2}{n}} = \sqrt{\frac{120}{5}} = \sqrt{24} = \mathbf{4.8989.}$$

Example 8. *Calculate S.D. for the following data:*

x	5	15	25	35	45	55
f	12	18	27	20	17	6

Solution. **Calculation of S.D.**

x	*f*	*fx*	$x - \bar{x}$	$(x - \bar{x})^2$	$f(x - \bar{x})^2$
5	12	60	– 23	529	6348
15	18	270	– 13	169	3042
25	27	675	– 3	9	243
35	20	700	7	49	980
45	17	765	17	289	4913
55	6	330	27	729	4374
	N = 100	$\Sigma fx = 2000$			$\Sigma f(x - \bar{x})^2 = 19900$

$$\bar{x} = \frac{\Sigma fx}{N} = \frac{2800}{100} = 28.$$

Now $$\text{S.D.} = \sqrt{\frac{\Sigma f(x - \bar{x})^2}{N}} = \sqrt{\frac{19900}{100}} = \sqrt{199} = \mathbf{14.1067.}$$

EXERCISE

1. Find the coeff. of Q.D. for the following distribution:

Marks	0–4	4–8	8–12	12–14
No. of students	10	12	18	7
Marks	14–18	18–20	20–25	25 and above
No. of students	5	8	4	6

2. Find the M.D. from A.M. for the following data:

x	3	5	7	9	11	13
f	*2*	*7*	*10*	*9*	*5*	*2*

3. Calculate S.D. for the following frequency distribution:

Class	***Frequency***	***Class***	***Frequency***
4–8	11	24–28	9
8–12	13	28–32	17
12–16	16	32–36	6
16–20	14	36–40	4
20–24	14		

PERIOD–II

M12CH6

TOPIC

6

Interpretation of Linear and Quadratic Graphs

6.1. GRAPHICAL METHOD TO SOLVE A PAIR OF LINEAR EQUATIONS IN TWO VARIABLES

Two linear equations in two variables form a system of linear equations. A solution to a system of linear equations is an ordered pair which satisfies both equations in the system.

For example, the ordered pair (3, 4) satisfies the system of equations

$$2x - 3y + 6 = 0$$

and $$2x - y - 2 = 0$$

since $$2 \times 3 - 3 \times 4 + 6 = 0$$

i.e., $$6 - 12 + 6 = 0$$

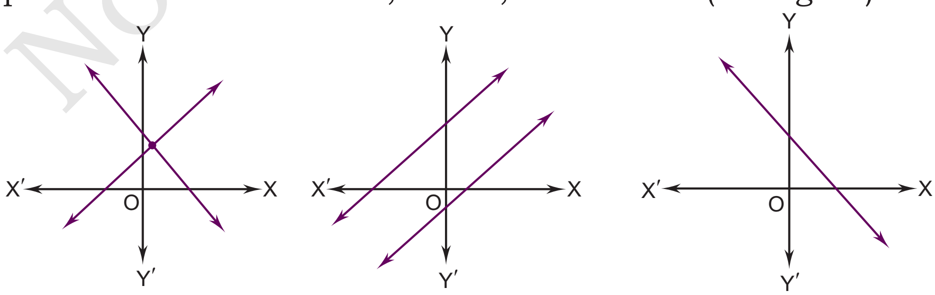
and $$2 \times 3 - 4 - 2 = 0 \quad \text{i.e.,} \quad 6 - 6 = 0$$

both are true statements.

Thus, $x = 3$ and $y = 4$ or the ordered pair (3, 4) is a solution of the given system of equations. The solution set to the system is {(3, 4)}.

The solution can be obtained by graphing both equations. The coordinates of the point of intersection give the solution of the system.

If the lines intersect, they will intersect in only one point giving a unique solution to the system (see figure). If the lines are parallel, there is no point of intersection and, hence, no solution (see figure).



Unique solution (intersecting lines) | No solution (parallel lines) | Infinitely many solutions (coincident lines)

If the lines are coincident, i.e., same line for both equations, then every point on the line is a common point and, hence, there are infinitely many solutions (see figure).

We shall concentrate on system having a unique solution.

Example 1. *Find the solution set of the following system of equations graphically:*

$$2x + 5y = 10 \quad \text{and} \quad x = -5$$

Solution. Given equations are:

$$2x + 5y = 10 \qquad \ldots(1) \qquad x = -5 \qquad \ldots(2)$$

From (1), $y = \dfrac{10 - 2r}{5}$

Table of values for (1)

x	0	5
y	2	0

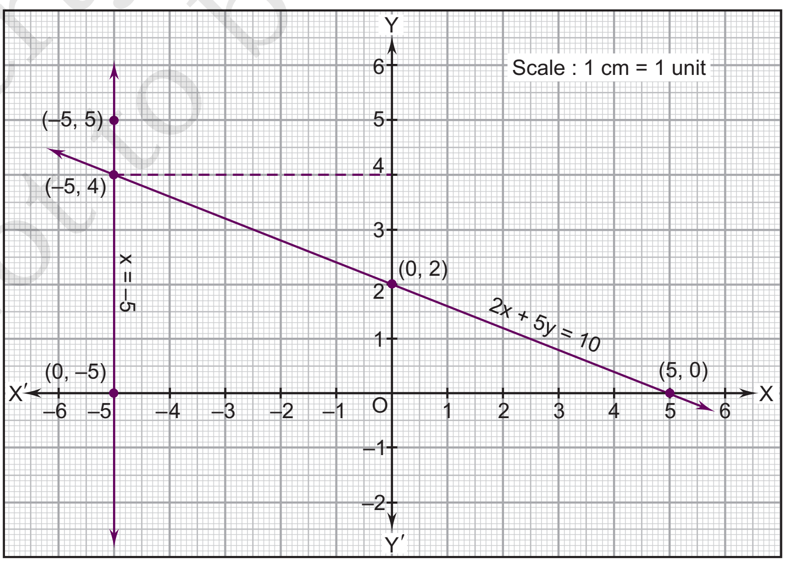
Table of values for (2)

x	–5	–5
y	0	5

Plot the ordered pairs (0, 2) and (5, 0). Join and produce both ways. This is the graph of equation (1).

Plot the ordered pairs (–5, 0) and (–5, 5). Join and produce both ways. This vertical line is the graph of equation (2).

The two lines intersect at a unique point (–5, 4) (Figure). Therefore, the ordered pair (–5, 4), i.e., $x = -5$, $y = 4$ is the solution of the system. The solution set $S = \{(-5, 4)\}$ is a singleton set.



6.2. NON-GRAPHICAL METHODS FOR SOLVING A SYSTEM OF LINEAR EQUATIONS IN TWO VARIABLES

(a) Elimination Method

Procedure:

(i) If necessary, re-write the equations so that the constant term in both equations is on the same side, left or right.

(ii) Multiply one or both equations, if necessary, by appropriate non-zero numbers so that addition or subtraction eliminates one variable. Now we have an equation in one variable, x or y.

(iii) Solve the resulting single variable equation for the variable involved.

(iv) Substitute the value obtained in step (iii) in either of the two original equations and solve for the other variable.

Example 2. *Solve the following system of equations by elimination method:*

$$4x + 3y - 14 = 0, \quad 9x - 5y = 55.$$

Solution. The given equations are

$$4x + 3y = 14 \quad ...(1) \quad \text{and} \quad 9x - 5y = 55 \quad ...(2)$$

(Re-write equation (1) so that the constant term is on same side in both equations.)

Let us eliminate *y*

Co-efficients of y in the two equations are 3 and 5 (numerically). LCM of 3 and 5 is 15. To make both co-efficients 15 (numerically), we multiply equation (1) by 5 and equation (2) by 3, we get

$$20x + 15y = 70 ...(3) \quad \text{and} \quad 27x - 15y = 165 \quad ...(4)$$

Adding (3) and (4), we get

$$47x = 235 \quad [y \text{ eliminated}]$$

$$\Rightarrow \quad x = \frac{5 \times 47}{47} = 5$$

Replacing x by 5 in equation (1), we get

$4 \times 5 + 3y = 14 \quad \Rightarrow \quad 3y = 14 - 20$

$$\Rightarrow \quad 3y = -6 \quad \Rightarrow \quad y = \frac{-6}{3} = -2$$

Therefore, the solution is $x = 5$, $y = -2$.

Hence, the solution set is $S = \{(5, -2)\}$.

(b) Substitution Method

Procedure:

(i) Solve one of the given equations for one variable in terms of the other, i.e., x in terms of y or y in terms of x.

(ii) Substitute this value of the variable in the other equation. This results into a single variable linear equation.

(iii) Solve the equation of step (ii)

(iv) Substitute this value in the expression obtained in step (i) to get the value of other variable.

Example 3. *Solve the following system of equations by substitution method:*

$$14x - 3y = 54, \quad 21x - 8y = 95.$$

Solution. The given equations are

$$14x - 3y = 54 \quad \ldots(1)$$

and

$$21x - 8y = 95 \quad \ldots(2)$$

Let us find *x* in terms of *y* from equation (1)

(We can find the same from equation (2) also.)

From (1), $14x = 54 + 3y$

or $x = \dfrac{54 + 3y}{14}$ ...(3)

Substituting this value of x in equation (2), we get

$$21\left(\frac{54 + 3y}{14}\right) - 8y = 95 \Rightarrow 3 \times 7\left(\frac{54 + 3y}{2 \times 7}\right) - 8y = 95$$

$$\Rightarrow \quad 3\left(\frac{54 + 3y}{2}\right) - 8y = 95$$

Multiplying both sides by 2, we get

$$3(54 + 3y) - 2(8y) = 2 \times 95$$

$$\Rightarrow \quad 162 + 9y - 16y = 190$$

$$\Rightarrow \quad -7y = 190 - 162$$

$$\Rightarrow \quad -7y = 28$$

$$\Rightarrow \quad y = \frac{4 \times 7}{-7} = -4$$

Substituting this value of y in equation (3), we get

$$x = \frac{54 + 3(-4)}{14} = \frac{54 - 12}{14} = \frac{42}{14} = 3$$

Therefore, the solution is $x = 3$, $y = -4$

Hence the solution set is $S = \{(3, -4)\}$.

6.3. GRAPHICAL METHOD TO SOLVE QUADRATIC EQUATION

An alternative method for solving a quadratic equation graphically is to find the intersection of the curve $y = x^2$ with a particular line.

Example 4. *Draw the graph of* $y = x^2$ *for* $-3 \le x \le 3$.

Use the graph to solve $x^2 + 2x - 3 = 0$.

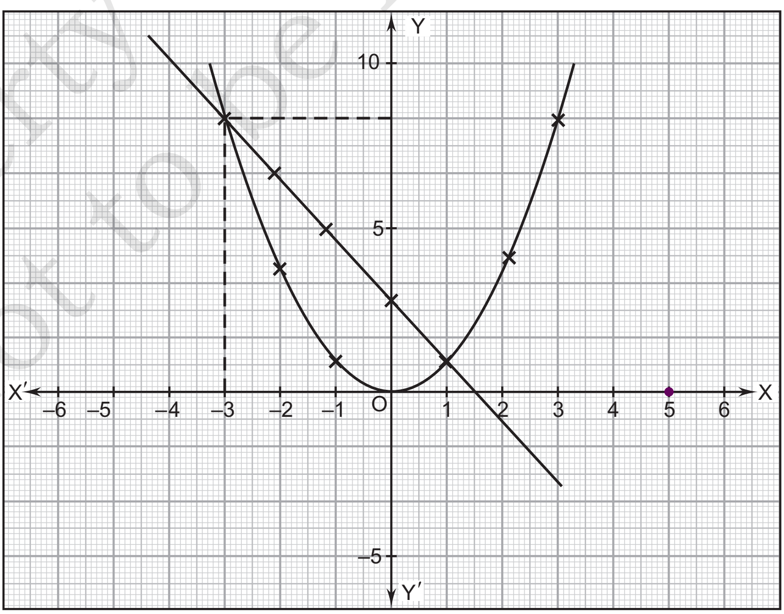
The equation $x^2 + 2x - 3 = 0$ *is the same as* $x^2 = -2x + 3$.

The table of values is:

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$-2x + 3$	9	7	5	3	1	-1	-3

Draw the graph of $y = x^2$ *and the line* $y = -2x + 3$ *on the same coordinate axes.*

Solution.



6.4. QUADRATIC FUNCTIONS AND EQUATIONS

Solving Quadratic Equations by Factorization

(a) If a, b and c are real numbers with $a \neq 0$, then the function

$$f(x) = ax^2 + bx + c$$

is called a **quadratic function**.

For example, $3x^2 - 5x + 2$, $\sqrt{2}x^2 - 2.5x + 1$ are quadratic functions.

Since, $f(x)$ represents a real number for any real value of x, the domain of a quadratic function is R, the set of all real numbers. The range of a quadratic function depends on the values of a, b and c.

(b) If a, b, and c are real numbers with $a \neq 0$, then the equation

$$ax^2 + bx + c = 0 \qquad \ldots(1)$$

is called a **quadratic equation** in **standard form**.

Clearly, if $f(x)$ is a quadratic function in x, then $f(x) = 0$ is a quadratic equation in x.

The values of x which satisfy equation (1) are called the **solutions** or **roots** or **truth values** of equation (1).

A real number α is said to satisfy equation (1) if $a\alpha^2 + b\alpha + c = 0$, i.e., if on replacing x by α in the left hand side, we get the right hand side (0).

Every quadratic equation has exactly two real roots if $b^2 - 4ac \geq 0$. The expression $b^2 - 4ac$ is called the **discriminant** of the quadratic equation.

If $b^2 - 4ac < 0$, then the quadratic equation has no real roots.

If $b^2 - 4ac > 0$, then the two roots are real and unequal.

If $b^2 - 4ac = 0$, then the two roots are real and equal.

The set of all truth values or solutions of a quadratic equation is called the truth set or the solution set of the quadratic equation. It is denoted by T or S. Thus,

$$T = \phi \quad \text{if } b^2 - 4ac < 0$$
$$T = \{\alpha\} \quad \text{if } b^2 - 4ac = 0$$
$$T = \{\alpha, \beta\} \quad \text{if } b^2 - 4ac > 0$$

A quadratic equation can be solved by a variety of methods

(i) by factorization

(ii) by quadratic formula

(iii) by the method of completing a square

(iv) by graph

Here, we discuss the first method only and assume that $b^2 - 4ac \geq 0$ so that the quadratic equation has two real roots.

Method

I. Write the given equation in the standard form $ax^2 + bx + c = 0$, by transposing all terms to the left hand side.

If a is negative, then multiply both sides by (–1), i.e., change the sign throughout so that a is positive.

II. Factorize the left hand side.

(i) If $b = 0$, use $X^2 - Y^2 = (X + Y)(X - Y)$

(ii) If $c = 0$, then $ax^2 + bx = 0 \quad \Rightarrow \quad x(ax + b) = 0$

(iii) If a, b and c are all non-zero, then find two numbers α and β such that $\alpha + \beta = b$ and $\alpha\beta = ac$.

In the given equation, replace b by $(\alpha + \beta)$, apply distributive property and group in pairs. Thus,

$$ax^2 + bx + c = 0$$

$$\Rightarrow \quad (px + q)(rx + s) = 0$$

III. Apply zero-product principle, which states that if the product of two algebraic expressions is zero, then at least one of the factors is equal to zero. Thus,

$$(px + q)(rx + s) = 0$$

$$\Rightarrow \quad px + q = 0 \quad \text{or} \quad rx + s = 0$$

$$\Rightarrow \quad x = -\frac{q}{p} \quad \text{or} \quad x = -\frac{s}{r}$$

IV. Check the solution in the original equation.

Hence, the truth set is $T = \left\{-\frac{q}{p}, -\frac{s}{r}\right\}$.

Example 5. *Solve by factorization:*

$$3x^2 + 7x = 0.$$

Solution. The given equation is

$$3x^2 + 7x = 0 \qquad \ldots(1)$$

(here $c = 0$)

or $\quad x(3x + 7) = 0$

By zero-product principle, we have

$$x = 0 \quad \text{or} \quad 3x + 7 = 0$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad 3x = -7$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = -\frac{7}{3}$$

Example 6. *Solve by factorization:*

$$2x^2 + 10 = 9x.$$

Solution. The given equation is

$$2x^2 + 10 = 9x \quad \text{...(1)}$$

Transposing all terms to left hand side

$$2x^2 - 9x + 10 = 0$$

Here $a = 2$, $b = -9$, $c = 10$

The two numbers whose sum is $b = -9$ and product is $ac = 2 \times 10 = 20$ are -5 and -4.

Replacing -9 by $-5 - 4$, we get

$$2x^2 - 5x - 4x + 10 = 0$$

$$\Rightarrow \quad x(2x - 5) - 2(2x - 5) = 0$$

$$\Rightarrow \quad (2x - 5)(x - 2) = 0$$

By zero-product principle

$$2x - 5 = 0 \quad \text{or} \quad x - 2 = 0$$

$$\Rightarrow \quad 2x = 5 \quad \text{or} \quad x = 2$$

$$\Rightarrow \quad x = \frac{5}{2} \quad \text{or} \quad x = 2$$

6.5. GRAPHICAL SOLUTION OF QUADRATIC EQUATIONS

The standard form of a quadratic equation is

$$ax^2 + bx + c = 0 \quad \text{...(1)}$$

where a, b and c are real numbers and $a \neq 0$.

If we write $$f(x) = ax^2 + bx + c \quad \text{...(2)}$$

then (2) is a quadratic function.

To solve the quadratic equation (1)graphically, we draw the graph of quadratic function (2), i.e., $y = ax^2 + bx + c$ where $y = f(x)$. The values of x for which $y = 0$, i.e., $ax^2 + bx + c = 0$ are the solutions of equation (1). On the graph of $y = f(x)$, $y = 0$ where the graph intersects x-axis.

If the graph

(i) does not intersect x-axis anywhere, then equation (1) has no solution or root or truth value.

The truth set $T = \phi$

(ii) touches x-axis at the point $(\alpha, 0)$, the $x = \alpha$ is the twice repeated solution of equation (1).

The truth set $T = \{\alpha\}$

(iii) intersects x-axis at two distinct points $(\alpha, 0)$ and $(\beta, 0)$, then $x = \alpha$ and $x = \beta$ are the two solutions of equation (1).

The truth set $T = \{\alpha, \beta\}$.

Graph of $f(x) = ax^2 + bx + c$

Let $y = f(x)$, then $y = ax^2 + bx + c$

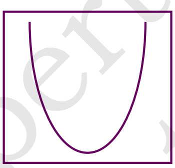
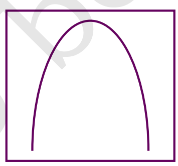
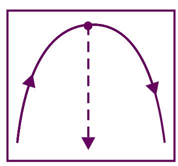
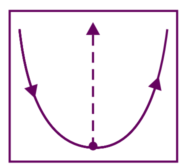
Put $x = x_1, x_2, \ldots, x_7$ and find the corresponding values of y. Display the values in the form of a table, called the table of values. Plot the seven ordered pairs (x, y) and pass a smooth curve through them. This curve is called a Parabola.

The graph of

$$y = ax^2 + bx + c, \; a \neq 0 \qquad \ldots(1)$$

is always a parabola.

When $a > 0$, the parabola opens upwards and is, therefore, called an upward parabola.

When $a < 0$, the parabola opens downwards and is, therefore, called Downward Parabola.

An important point in the graph of a parabola is the point where the graph takes a turn. This point is called the **vertex** of the parabola.

In an upward parabola, vertex is the lowest point. On the left of the vertex, the graph is falling and on the right of the vertex, the graph is rising.

In a downward parabola the vertex is the highest point. On the left of the vertex, the graph is rising and on the right of the vertex, the graph is falling.

The x-coorrdinate of the vertex is given by $-\frac{b}{2a}$. The y-coordinate can be obtained by putting $x = -\frac{b}{2a}$ in equation (1).

The line drawn vertically upwards through the vertex of an upward parabola is called the **axis** of the parabola. It is the line of symmetry for the upward parabola. Folding the graph sheet about this line, the two halves of the parabola coincide.

The line drawn vertically downwards through the vertex of a downward parabola is called the axis of the parabola. It is the line of symmetry for the downward parabola. Folding the graph sheet about this line, the two halves of the parabola coincide.

Steps for solving $ax^2 + bx + c = 0$, $a \neq 0$, graphically.

1. Let $y = ax^2 + bx + c$

2. Check the sign of a.

$a > 0 \quad \Rightarrow$ upward parabola

$a < 0 \quad \Rightarrow$ downward parabola

3. Find $-\frac{b}{2a}$. Put $x = -\frac{b}{2a}$ in equation (1) and find y. This gives the vertex of parabola, say (x_0, y_0).

4. Form the table of values. Give six integral values of x, three less than x_0 and three more than x_0. This gives seven ordered pairs (x, y).

5. Plot the seven ordered pairs.

6. Passasmoothcurvethroughalltheplottedpoints.Thisisthegraphof $f(x) = ax^2 + bx + c$.

7. Mark the points where the graph intersects x-axis and find the ordered pairs corresponding to them.

8. The x-coordinates of points in step (7) are the required solutions or truth values or roots of the quadratic equation $ax^2 + bx + c = 0$.

Example 7. *Draw the graph of $f(x) = x^2 - 2x + 2$ and hence solve the equation $x^2 - 2x + 2 = 0$.*

Solution. Let $y = x^2 - 2x + 2$ ...(1)

Here $a = 1$, $b = -2$, $c = 2$

Since $a > 0$, the graph of (1) is an upward parabola.

$$-\frac{b}{2a} = -\frac{-2}{2 \times 1} = 1$$

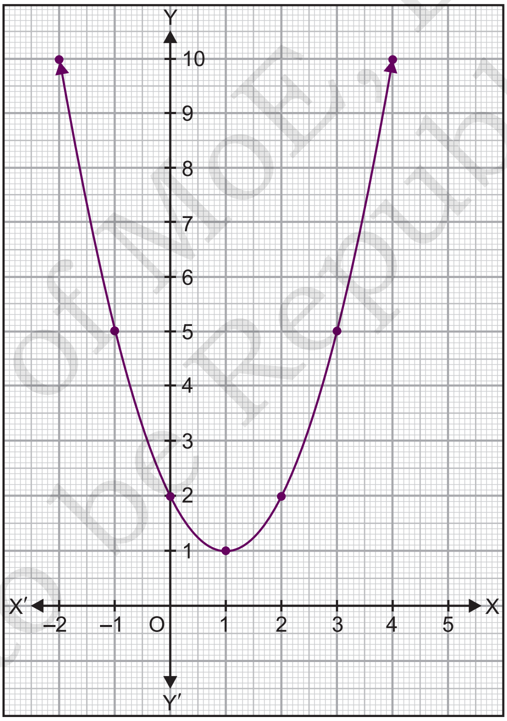
Putting $x = 1$ in (1), $y = 1^2 - 2 \times 1 + 2 = 1$

$\Rightarrow$ (1, 1) is the vertex.

Table of values $y = x^2 - 2x + 2$

x	–2	–1	0	1	2	3	4
y	10	5	2	1	2	5	10

Plot the seven ordered pairs. Pass a smooth curve through all the plotted ordered pairs. This is the graph of $y = x^2 - 2x + 2$.



This is an upward parabola. The arrows at the two ends indicate that the parabola extends indefinitely in both directions. Since the graph does not intersect x-axis anywhere, the quadratic equation $x^2 - 2x + 2 = 0$ has no truth value and the truth set $T = \phi$.

Example 8. *Draw the graph of $f(x) = -4x^2 - 12x - 9$ and, hence, solve the equation $4x^2 + 12x + 9 = 0$.*

Solution. Let $y = -4x^2 - 12x - 9$ ...(1)

Here $a = -4$, $b = -12$, $c = -9$

Since $a < 0$, the graph of (1) is a downward parabola.

$$-\frac{b}{2a} = -\frac{-12}{2(-4)} = -\frac{3}{2}$$

Putting $x = -\frac{3}{2}$ in (1),

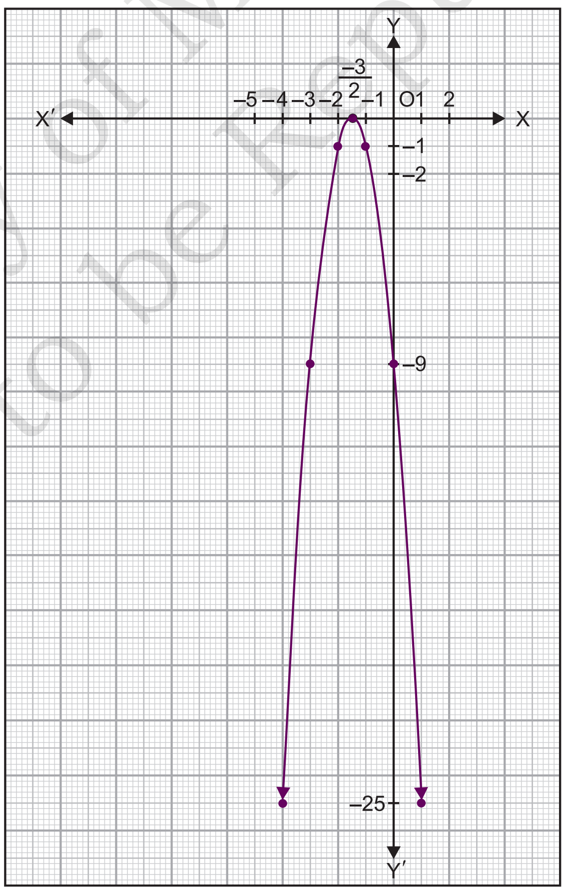
$$y = -4\left(-\frac{3}{2}\right)^2 - 12\left(-\frac{3}{2}\right) - 9$$

$$= -4\left(\frac{9}{4}\right) + 18 - 9 = -9 + 9 = 0$$

$\Rightarrow \left(-\frac{3}{2}, 0\right)$ is the vertex or turning point of (1).

Table of values for $y = -4x^2 - 12x - 9$

x	–4	–3	–2	–3/2	–1	0	1
y	–25	–9	–1	0	–1	–9	–25



Plot the seven ordered pairs. Pass a smooth curve through all the plotted ordered pairs. This is the graph of $y = -4x^2 - 12x - 9$. This is a downward parabola. The arrow at the two ends indicates that the parabola extends indefinitely in both directions.

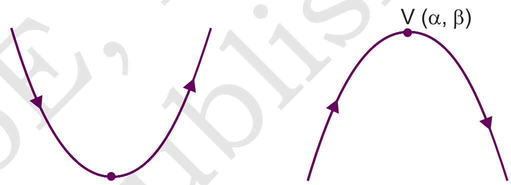
Since the graph touches x-axis at $\left(-\frac{3}{2}, 0\right)$, $x = -\frac{3}{2}$ is the only solution of the quadratic equation

$$-4x^2 - 12x - 9 = 0 \quad \text{or} \quad 4x^2 + 12x + 9 = 0$$

Hence, the truth set $T = \{-3/2\}$

6.6. INCREASING/DECREASING VALUES OF QUADRATIC GRAPHS

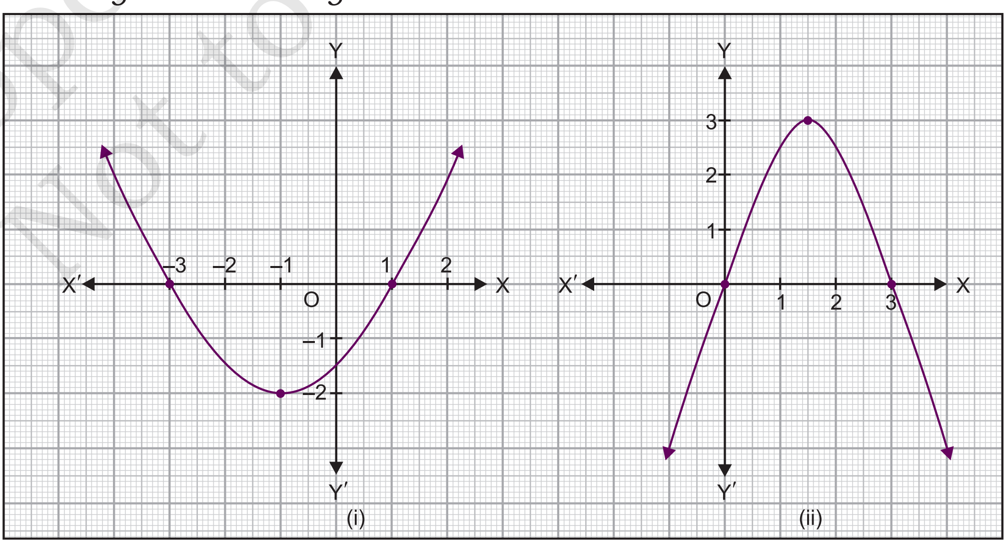
We know that the graph of a quadratic function is a parabola, upward or downward. Vertex of a parabola is a turning point. Let $V(\alpha, \beta)$ be the vertex of a parabola.



In an upward parabola, the graph is falling on the left of V and rising on the right of V. Thus, $y = f(x)$ is decreasing for $-\infty < x < \alpha$ and increasing for $\alpha < x < \infty$.

In a downward parabola, the graph is rising on the left of V and falling on the right of V. Thus, $y = f(x)$ is increasing for $-\infty < x < \alpha$ and decreasing for $\alpha < x < \infty$.

Example 9. *Find the range of values of x for which the following graphs are increasing or decreasing.*



Solution. In figure (i), the vertex is $V(-1, -2)$ so that $\alpha = -1$. The parabola opens upwards. The graph is falling on the left of V and rising on the right of V.

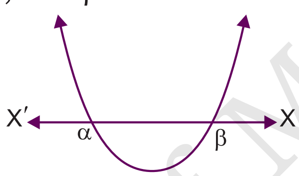
Thus, $y = f(x)$ is decreasing for $-\infty < x < -1$ and increasing for $-1 < x < \infty$.

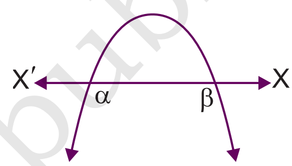
In Figure (ii), the vertex is $V(1.5, 3)$ so that $\alpha = 1.5$. The parabola opens downwards. The graph is rising on the left of V and falling on the right of V.

Thus, $y = f(x)$ is increasing for $-\infty < x < 1.5$ and decreasing for $1.5 < x < \infty$.

6.7. POSITIVE/NEGATIVE VALUES OF A QUADRATIC GRAPH

(a) Let a quadratic graph intersect x-axis at points whose x-coordinates are α and β, $\alpha < \beta$.





If the graph is an upward parabola, figure, then the graph is

(i) above x-axis, i.e., y is positive for $x < \alpha$ and $x > \beta$.

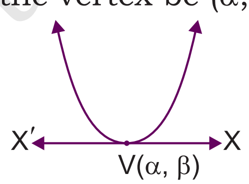
(ii) below x-axis, i.e., y is negative for $\alpha < x < \beta$.

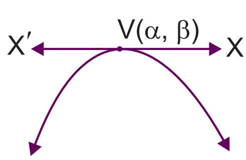
If the graph is a downward parabola, figure, then the graph is

(i) above x-axis, i.e., y is positive for $\alpha < x < \beta$

(ii) below x-axis, i.e., y is negative for $x < \alpha$ and $x > \beta$.

(b) Suppose a quadratic graph touches x-axis, then the point of contact is the vertex. Let the vertex be (α, β).

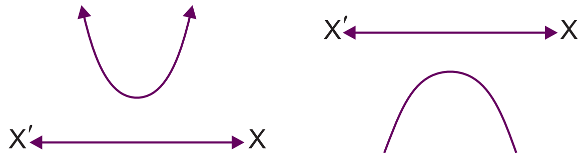




If the graph is an upward parabola, Fig. 3, then the graph is above x-axis except at the vertex. Thus y is positive for all real values of x except 'α', i.e., for $R - \{\alpha\}$.

If the graph is a downward parabola, figure, then the graph is below x-axis except at the vertex. Thus, y is negative for all real values of x except 'α', i.e., for $R - \{\alpha\}$.

(c) Suppose a quadratic graph does not intersect x-axis anywhere.



If the graph is an upward parabola, figure, then the graph is entirely above x-axis. Thus, y is positive for all $x \in R$, i.e., for $-\infty < x < \infty$.

If the graph is a downward parabola, figure, then the graph is entirely below x-axis. Thus, y is negative for all $x \in R$, i.e., $-\infty < x < \infty$.

Example 10. *Find the range of values of x for which $y = x^2 - 4x + 3$ is positive or negative.*

Solution. Given $y = x^2 - 4x + 3$

Here, $a = 1, b = -4, c = 3$

Since $a > 0$, the parabola opens upwards.

The parabola intersects x-axis where $y = 0$

i.e., $x^2 - 4x + 3 = 0$ or $(x - 1)(x - 3) = 0$

$\Rightarrow$ $x = 1$ or 3

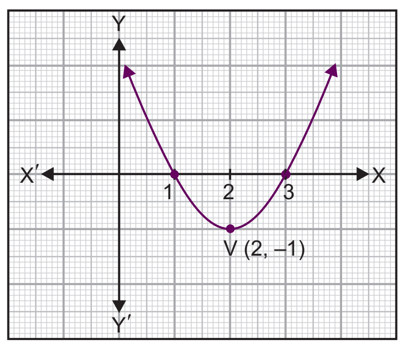
$$-\frac{b}{2a} = -\frac{-4}{2 \times 1} = 2$$

Putting $x = 2$, $y = 2^2 - 4 \times 2 + 3 = -1$

$\Rightarrow$ Vertex is $(2, -1)$.

Therefore, y is positive, i.e., the graphs is above x-axis when $x < 1$ or $x > 3$.

y is negative, i.e., the graph is below x-axis when $1 < x < 3$.



EXERCISE

1. Find the solution set of the following system of equations graphically:

$$2x - y - 1 = 0 \quad \text{and} \quad x - 2y + 1 = 0.$$

2. Solve the factorization $4x^2 - 49 = 0$.
3. Solve the factorization $9x^2 - 30x + 25 = 0$.
4. Solve $x^2 + 2x - 8 = 0$ graphically.

PERIOD–II

M12CH7

TOPIC

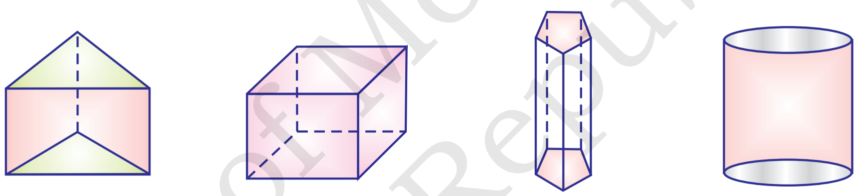
7

Mensuration-II

7.1. SURFACE AREA OF A PRISM

Prisms are solid with uniform cross-section. The cross-section may be triangular, rectangular, circular of shapes. A prism is named by the spade of its cross-section.

For example:



Triangular prism   Rectangular prism   Pentagonal prism   Cylinder

7.2. CUBOID

It is a prism with rectangular faces. All the sides of a cuboid are rectangles.

Formula used to calculate surface area of a cuboid is:

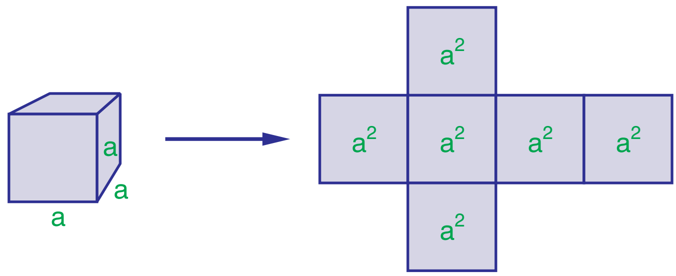
Surface area of a cuboid = $2(lb + bh + hl)$

where l, b, and h are respectively the three edges of the cuboid.

7.3. CUBE

Recall that a cuboid having equal length, equal length, breadth and height is called a cube. If each edge of the cube is 'a', the surface area of this cube would be

$2(a \times a + a \times a + a \times a)$ *i.e.*, (see figure)



Formula used to calculate surface area of cuboid is

Surface area of a cube = $6a^2$

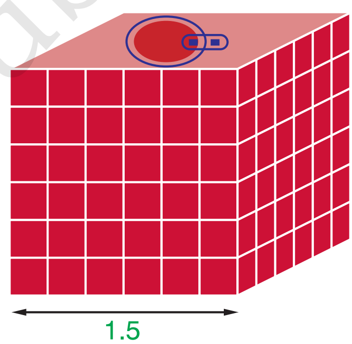
Example 1. *The length, breadth and height of a cuboid are 15 cm, 10 cm and 20 cm respectively. Find the surface area.*

Solution. Surface area of a cuboid = $2(lb + bh + hl)$

So, Surface area = 2[(15 × 10) + (10 × 20) + (20 × 15)] cm^2

= 2(150 × 200 + 300)cm^2 = 2 × 650 cm^2 = **1,300 cm^2**

Example 2. *Runesha has built a cubical water tank with lid for his house, with each outer edge 1.5 m long. He gets the outer walls of the tank covered with square tiles of side 25 cm (see figure) Find how much he will spend for the tiles, if the cost of the tiles is 3600* L$ *per dozen.*



Solution. Runesha is getting the four outer walls of the tank covered with tiles. He needs to calculate the surface area of the walls, to know the number of tiles required.

Edge of the cubical tank = 1.5 m = 150 cm

So, Surface area of the tank = 4 × 150 × 150 cm^2

Area of each square tile = Side × Side = 25 × 25 cm^2

So, the number of tiles required

$$= \frac{\text{Surface area of the tank}}{\text{Area of each tile}} = \frac{4 \times 150 \times 150}{25 \times 25} = 144$$

Cost of 1 dozen tiles, i.e., 12 tiles = 3,600 L$

Therefore, the cost of one tile = $\frac{3,600}{12}$ L$ = 300 L$

So, the cost of 144 tiles = 144 × 300 L$ = 43,200 L$

7.4. VOLUME OF A CUBOID

Solid objects occupy space. The measure of this occupied space is called the Volume of the object.

Formula used to calculate volume of a cuboid

$$= \text{length} \times \text{breadth} \times \text{height}$$

$$\Rightarrow \qquad V = l \times b \times h$$

Example 3. *What is the volume of a rectangular solid with edge 9 m, 3.5 m and 2.8 m?*

Solution. Volume = $l \times b \times h$ = 9 m × 3.5 m × 2.8 m = 88.2 m^3

Example 4. *A rectangular box has a volume of 9 m^3 and a base 50 cm by 25 cm. Find its height.*

Solution. Dimensions of the box are:

Volume (V) = 9 m^3; length (l) = 50 cm = 0.5 m; breadth (b) = 25 cm = 0.25 m

Since, volume = $l \times b \times h$

So, $$\text{height } (h) = \frac{\text{Volume}}{l \times b}$$

$$h = \frac{9 \text{ m}^3}{0.5 \text{ m} \times 0.25 \text{ m}} = 72 \text{ m}$$

$\therefore$ h = **72 m**

7.5. SURFACE AREA OF A RIGHT CIRCULAR CONE

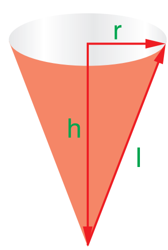
Formula used to calculate curved surface area of a cone is where r is its base radius and l is its slant height.

$$\text{Curved surface area of cone} = \frac{1}{2} \times l \times 2\pi r = \pi r l$$

Now, suppose h is the height of the cone. Then using Pythagoras theorem we have, $l^2 = r^2 + h^2$.

Therefore, $l = \sqrt{r^2 + h^2}$

If the base of the cone is closed, then a circular piece of paper of radius r is also required whose area is πr^2.



So, Formula used to calculate Total surface area of a cone is

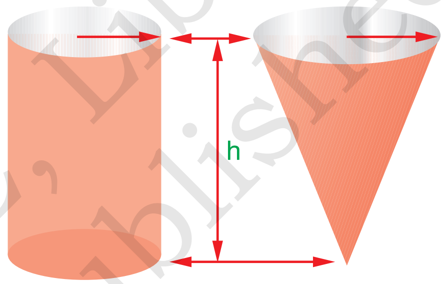
Total surface area of cone = $\pi rl + \pi r^2 = \pi r(l + r)$

Example 5. *The slant height of a cone is 10 cm and the base radius is 7 cm. Find the curved surface area.*

Solution. Curved surface area = $\pi rl = \frac{22}{7} \times 7 \times 10 \text{ cm}^2$ = **220 cm²**

7.6. VOLUME OF A RIGHT CIRCULAR CONE

Observe figure comprising a right circular cylinder and a right circular cone. They have the same base radius and the same height.



Formula used to calculate Volume of cone is

$$\text{Volume of a cone} = \frac{1}{3}\pi r^2 h$$

where r is the base radius and h is the height of the cone.

Example 6. *The height and the slant height of a cone are 21 cm and 28 cm respectively. Find the volume of the cone.*

Solution. By Pythagorean theorem $l^2 = r^2 + h^2$

$$\Rightarrow \qquad r = \sqrt{l^2 - h^2} - \sqrt{28^2 - 21^2} \text{ cm} = 7\sqrt{7}\text{cm}$$

So, volume of the cone = $\frac{1}{3}\pi r^2 h$

$$= \frac{1}{3} \times \frac{22}{7} \times 7\sqrt{7} \times 7\sqrt{7} \times 21 \text{ cm}^3 = \mathbf{7{,}546 \text{ cm}^3}$$

7.7. SURFACE AREA OF A PYRAMID

A pyramid has sides that are triangular-faced and a base. The base can be of any shape.

The lateral faces of a pyramid are always triangles. So, we use the area formula of a triangle to calculate the area of the lateral faces.

Area of each lateral face = $\frac{1}{2}$ × breadth × height = × l × h

[Here, breadth = side of square base (l)]

There are 4 lateral faces, so, total area of lateral faces

$$= 4 \times \frac{1}{2} lh = 2\ lh$$

Here, the base of pyramid is a square.

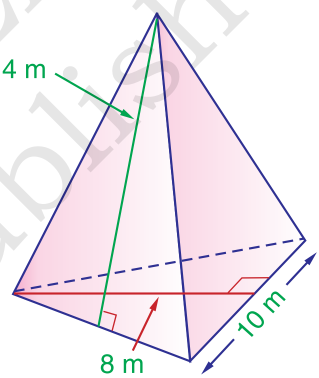
So, area of the base = l^2

Therefore, finally the sum of the area of the base and area of each lateral face is the area of the pyramid.

$$S = \text{area of base} + \text{areas of lateral faces}$$

So, $S = l^2 + 2lh$

Example 7. *Find the surface area of the regular pyramid shown in figure.*

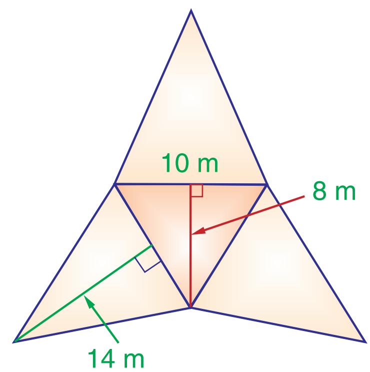


Solution. To help visualize the surface area clearly, sketch a net. Then use the net to find the area of base and the area of each lateral face.

Area of base, $A = \frac{1}{2} \times 10 \times 8 = 40\ m^2$

Area of each lateral face,

$$A = \frac{1}{2} \times 10 \times 14 = 70\ m^2$$



Finally, the surface area of the regular pyramid,

$$S = \text{area of base} + \text{areas of lateral faces}$$

$$= \underbrace{70 + 70 + 70}$$

$$= \mathbf{250\ m^2}$$

The pyramid has three congruent lateral faces. Count the area times.

The surface area is 250 square metre.

7.8. VOLUME AREA OF A PYRAMID

The volume of a pyramid can be calculated using this formula

$$\text{Volume of the pyramid} = \frac{1}{3}\text{area of the base} \times \text{height}$$

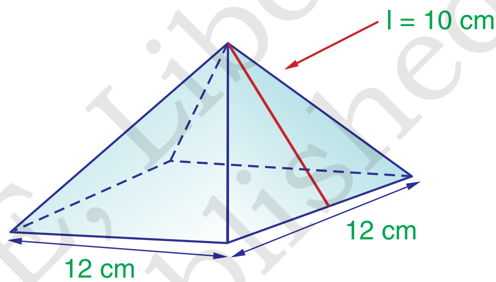
Example 8. *Find the volume of the square pyramid of perpendicular height 18.90 cm and the length of the side of base 3 cm.*

Solution. Here height (h) = 18.90 cm, and side (a) = 3 cm

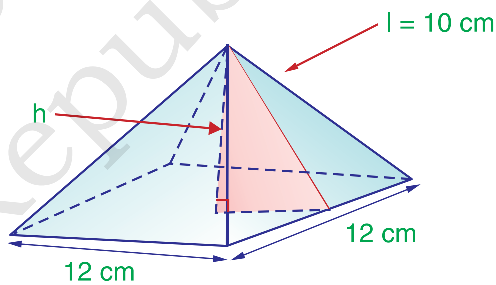
$$\therefore \quad \text{Volume of the pyramid} = \frac{1}{3} \text{ area of the base} \times \text{height}$$

$$= \frac{1}{3} a^2 h = (3)^2 \times 18.90 \text{ cm}^3 = \mathbf{56.7\ cm^3}$$

Example 9. *Find the volume of the following regular pyramids.*



Solution. We first need to find the height 'h', of the pyramid. We draw an altitude to the base of the prism and form a right triangle as shown in figure. We can then use the Pythagorean Theorem to find h.



$$\Rightarrow \quad 6^2 + h^2 = 10^2$$

$$\Rightarrow \quad 36 + h^2 = 100$$

$$\Rightarrow \quad h^2 = 64$$

$$\Rightarrow \quad h = 8 \text{ cm}$$

The base of the pyramid is a square with area (a^2) = 12^2 cm^2 = 144 cm^2.

The height of the pyramid (h) = 8 cm.

Therefore, the volume, $V = \frac{1}{3} a^2 h = \frac{1}{3}$ (144) (8) cm^3 = 384 cm^3.

7.9. SURFACE AREA OF A SPHERE

A sphere is a three dimensional circular figure. All points on its surface are equidistant from its centre.

Formula used to calculate surface area of a sphere is

Surface area of a sphere = $4\pi r^2$ where r is the radius of the sphere.

Example 10. *Find the surface area of a sphere of radius 7 cm.*

Solution. The surface area of a sphere of radius 7 cm would be

$$4\pi r^2 = 4 \times \frac{22}{7} \times 7 \times 7 \text{ cm}^2 = \mathbf{616\ cm^2}$$

Formula used to calculate the volume of a sphere is

Volume of a sphere = $\frac{4}{3}\pi r^3$

where r is the radius of the sphere.

Example 11. *Find the volume of a sphere of radius 11.2 cm.*

Solution. Required volume = $\frac{4}{3}\pi r^3 = \frac{4}{3}\times\frac{22}{7}\times 11.2\times 11.2\times 11.2\text{ cm}^3$

$= 5{,}887.32\text{ cm}^3$

A Great circle is any circle of maximum diameter that can be on surface of a sphere.

The only great circle perpendicular to the axis of the earth is the **equator**. The earth's axis runs from the North pole to the South pole.

Latitude and Longitude: We can represent a point $P(x, y)$ on the surface of earth and the coordinates of this point are known as **geographic coordinates**. These coordinates are measured in degrees and represent angular distance calculated from the centre of the earth.

An imaginary line passing through the centre of the earth from East to West is known as **equator**.

Let P be any point on the equator. Draw a circle around the whole earth so that it goes through the point on the equator and the North and South poles. The line so drawn is known as **line of longitude**.

The line of longitude that goes through 0° is known as **prime meridian**. (Historically, the position for 0° was chosen at **Greenwich** in England. Also,

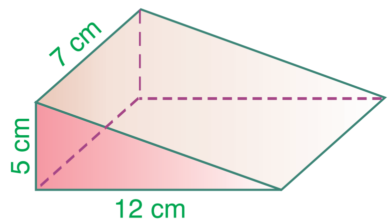
A **longitude** is an angle ϕ_1 measured East or West of the prime meridian satisfying $0° \le \phi_1 \le 180°$.

Note: 0°E = 0°W = 0°; 180°E = 180°W = 180°

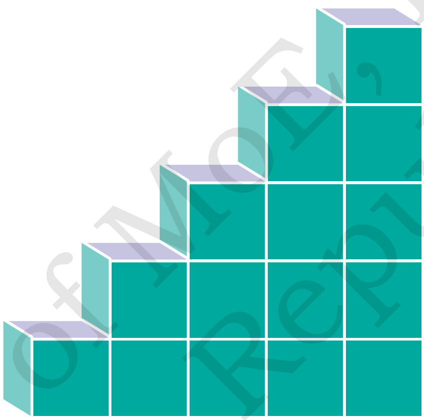
A **latitude** is an angle ϕ_2 measured North or South of the equator satisfying $0° \le \phi_2 \le 90°$. Any place on the earth can be represented by the geographical coordinate P(x, y) where x is the latitude measure and y is the longitude measure of that place shown in figure.

EXERCISE

1. What is the total surface area of this prism?

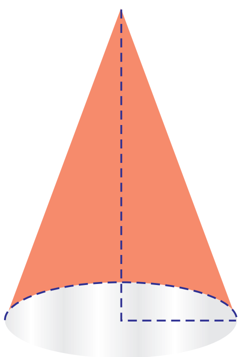


2. A child is playing with building blocks, which are of the shape of cubes. It has built a structure as shown in figure. The edge of each cube is 3 cm. Find the volume of the structure built by the child.

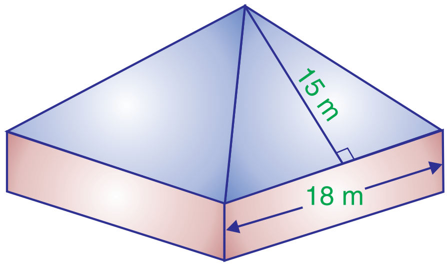


3. The height of a cone is 16 cm and its base radius is 12 cm. Find the curved surface area and the total surface area of the cone. (Use π = 3.14).

4. Find the volume of a cone (see figure) with a radius of 7 cm and a height of 30 cm.



5. The roof is shaped like a square pyramid. One bundle of shingles covers 20 square metre. How many bundles should you buy to cover the roof?



6. Identify the major cities closest to the following locations. Use Liberia latitude and longitude map.

(6.2°N, 10.1°W)

M12CH8

TOPIC

8

Logical Reasoning

8.1. STATEMENTS

We all know that human beings can think more logically as compared to other species, i.e., animals or birds. This ability makes them far more superior to other species. Logic as language of Mathematics is the study of general pattern of reasoning. In this unit, we shall study some basics of logical reasoning.

We express our ideas by means of sentences. A sentence can be (i) true (ii) false (iii) both.

Now, consider the following sentences:

(i) 5 is less than 7

(ii) 5 is greater than 8

(iii) Mathematics is an interesting subject.

Without any confusion, we can clearly decide that (i) is true and (ii) is false. What about (iii)? Some students may agree to it and call it true while others may disagree and call it false. The sentence is ambiguous. We cannot say whether it is always true or false.

A sentence which is either true or false but not both is called a 'logical statement' or 'a mathematically acceptable statement' or briefly 'a statement'.

In the above example, (i) and (ii) are statements whereas (iii) is not a statement.

In logical reasoning, the basic unit involved is a statement.

A true sentence is also called a **valid statement** and a false sentence is also called an **invalid statement**. A sentence which is both true and false simultaneously is called a **paradox**. Every statement is a sentence, true or false, but every sentence need not be a statement. Statements are denoted by small letters p, q, r, ...

For example, p : 7 is a prime number

$q : \sqrt{2}$ is a rational number

In the grammatical sense, a statement is a declarative or assertive sentence. It is neither imperative nor interogative, nor optative, nor exclamatory.

For example, consider the following sentences:

(a) (i) 2012 was a leap year.

(ii) π is an irrational number.

(iii) Every square is a rectangle.

Each of these sentences is a declaration or an assertion. Each of them is true and, hence, each of them is a valid statement.

(b) (i) All prime numbers are odd.

(ii) $\sqrt{2}$ is a rational number.

(iii) Every set is a finite set.

Each of these sentences is a declaration or an assertion. Each of them is false and, hence, each of them is an invalid statement.

(c) (i) Switch on the light.

(ii) Please open the door.

(iii) Get out.

Each of the above sentences is a command or a request and, hence, an imperative sentence. None of them can be called true or false. Hence, none of them is a statement.

(d) (i) How do you do?

(ii) Where are you going?

(iii) When will you wake up?

Each of the above sentences is a question and, hence, an interogative sentence. None of them can be called true or false. Hence, none of them is a statement.

(e) (i) May you live long!

(ii) May God bless you!

(iii) Good morning every body!

Each of the above sentences is a wish or desire and, hence, an optative sentence. None of them can be called true or false. Hence none of them is a statement.

(f) (i) How beautiful is the rainbow!

(ii) Hurrah! I have passed with distinction.

(iii) What a fragrance!

Each of the above sentences is exclamatory. None of them can be called true or false. Hence, none of them is a statement.

8.2. NEGATION OF STATEMENTS

The denial of a statement is called the negation of the statement.

If p is a statement, then the negation of p is also a statement and is denoted by $\sim p$. It is read as 'not p'.

Let p : 7 is a prime number

be a given statement. Now consider the following statements:

(i) 7 is not a prime number.

(ii) It is false that 7 is a prime number.

(iii) It is not true that 7 is a prime number.

(iv) It is not the case that 7 is a prime number.

Each of these statements is opposite in meaning to the given statement. Hence each of these statements is the negation of p, i.e., each of these statements is $\sim p$.

Thus, negation of a statement p is formed by inserting the word 'not' if absent and by dropping the word 'not' if present. The negation of p is also formed by writting 'It is false that' or 'It is not true that' or It is not the case that' before p.

The truth value of $\sim p$ is always the opposite of the truth value of p. Negation changes a true statement into a false statement and a false statement into a true statement. In other words, if p is true, then $\sim p$ is false and if p is false, then $\sim p$ is true.

Example 1. *Negate the following statements:*

(i) *Kofi is not a lazy boy.* (ii) $\sqrt{7}$ *is a rational number.*

Solution. (i) Let p : Kofi is not a lazy boy.

Then $\sim p$: Kofi is a lazy boy.

(obtained by dropping 'not')

Note: The negation of p may also be written as:

$\sim p$: **It is false that** Kofi is $\underbrace{\text{not a lazy boy}}_{p}$.

or $\sim p$: **It is not the case that** Kofi is $\underbrace{\text{not a lazy boy}}_{p}$.

(ii) Let p : $\sqrt{7}$ is a rational number.

Then ~ p : $\sqrt{7}$ is not a rational number (obtained by inserting 'not')

Note: The negation of p may also be written as:
~ p : It is false to say that $\sqrt{7}$ is a rational number.
or ~ p : It is not the case that $\sqrt{7}$ is a rational number.
or ~ p : $\sqrt{7}$ is an irrational (not rational)number.
Clearly, p is false and ~ p is true.

8.3. IMPLICATIONS ($\Rightarrow$, $\Leftrightarrow$)

(a) 'If-then' Implication or Conditional Statement

In Mathematics and in our day-to-day life, expressions of the form 'if p', then q' occur very often.

For example:

(i) If $x = 4$, then $x^2 = 16$
(ii) If $3x - 2 = 10$, then $x = 4$
(iii) If $l \parallel m$ and $m \parallel n$, then $l \parallel n$.
(iv) If it rains, then I will not go out for a walk.

If p and q are two statements, then the statement 'if p, then q' is called an if, then' implication or simply an **implication or a conditional statement**. It is denoted by $p \Rightarrow q$ and read as 'p implies q.' Here p is called the **antecedent or hypothesis** and q is called the **consequent or conclusion**.

In example (i) above, let $p : x = 4$, $q : x^2 = 16$
then the symbolic form of statement (i) is $p \Rightarrow q$

In example (ii) above, let $p : 3x - 2 = 10$, $q : x = 4$
then the symbolic form of statement (ii) is $p \Rightarrow q$. Similarly, for (iii) and (iv)

It is very important to observe that if p is true, then q must be true, i.e., whenever the hypothesis holds, the conclusion must hold. However, when p is false, then q may be true or false, i.e., no restriction on q.

Consider p : You are born in some country

q : You are a citizen of that country

then $p \Rightarrow q$ is the statement:

If you are born in some country, then you are a citizen of that country.

Clearly, if p is true then q is true. What happens when p is false? i.e., when you are not born in a county, then you are not a born citizen yet you can acquire citizenship.

It may be noted that '$\Rightarrow$' is **not commutative**. Thus $p \Rightarrow q$ is different from $q \Rightarrow p$.

$p \Rightarrow q$ means p is sufficient for q

while $q \Rightarrow p$ means p is necessary for q.

(b) 'If and only if' Implication or Double Implication or Biconditional Statement

Let p and q be two statements such that $p \Rightarrow q$ and $q \Rightarrow p$ i.e., 'if p then q' and 'if q then p', then this compound statement is called 'if and only if' implication or double implication or biconditional statement. It is denoted by $p \Leftrightarrow q$ and read as 'p if and only if q'. For brevity, if and only if is written as 'iff'. Thus, $p \Leftrightarrow q$ or p iff q is a double implication or a biconditional statement.

Note that '$\Leftrightarrow$' is commutative. Thus, $p \Leftrightarrow q$ is same as $q \Leftrightarrow p$. Therefore, $p \Leftrightarrow q$ means p is necessary and sufficient for q.

For example:

(i) Since $3x - 2 = 10 \Rightarrow x = 4$

and $x = 4 \Rightarrow 3x - 2 = 10$

we say $3x - 2 = 10 \Leftrightarrow x = 4$

(ii) Let p : a triangle is equilateral

q : a triangle is equiangular

then $p \Rightarrow q$ is the statement 'If a triangle is equilateral then it is equiangular.

$q \Rightarrow p$ is the statement 'If a triangle is equiangular then it is equilateral'.

$p \Leftrightarrow q$ is the statement 'A triangle is equilateral, if and only if it is equiangular.'

8.4. USE OF VENN DIAGRAMS IN TESTING THE VALIDITY OF IMPLICATIONS

An **argument** is an assertion that a statement S follows from certain other statements $S_1, S_2, \ldots, S_n$.

The statement S is called the **Conclusion** and the statements $S_1, S_2, \ldots, S_n$ are called **hypothesis** or **premisis**.

An argument consisting of hypothesis $S_1, S_2, \ldots, S_n$ and conclusion S is said to be **valid** if S is true whenever all $S_1, S_2, \ldots, S_n$ are true, i.e., if $S_1, S_2, \ldots, S_n$ are all true $\Rightarrow$ S is true.

The validity of an argument can be tested by using Venn diagrams as follows:

(i) Represent the truth of the hypothesis by Venn diagrams.

(ii) Analyse the Venn diagrams to see whether they necessarily represent the truth of the conclusion. If so, then the argument is valid, otherwise it is invalid.

Example 2. *Use Venn diagrams to examine the validity of the following arguments:*

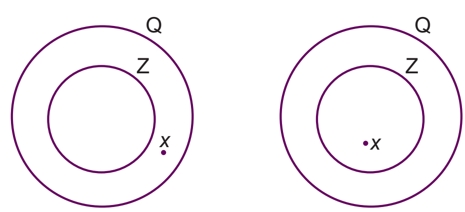
(i) S_1 : *All integers are rational numbers.*
S_2 : *x is a rational number.*
S : *x is an integer.*

(ii) S_1 : *All integers are rational numbers.*
S_2 : *x is a rational number.*
S : *x is not an integer.*

(iii) S_1 : *All integers are rational numbers.*
S_2 : *x is not a rational number.*
S : *x is not an integer.*

Solution. The three given statements in each part constitute an argument in which S_1, and S_2, are hypothesis and S is the conclusion.

Let Q denote the set of all rational numbers and let Z denotes the set of all integers. The truth of the statement S_1, $Z \subset Q$, is represented by placing the set Z entirely inside the set Q. The truth of the statement S_2 is represented by placing a dot labelled x inside the set Q. But the position of the dot with respect to the set Z is not known.



(a) (b)

x may be $\frac{2}{3}$ which is not an integer and hence, as shown in Fig. 1(a) the dot is outside the set *Z*.

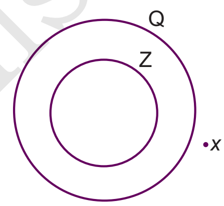
x may be –5 which is an integer and hence, as shown in Fig. 1 (b), the dot is inside the set *Z*.

Both the positions of *x* represent the truth of the statement S_2.

(i) The truth of the conclusion *S* that *x* is an integer does not necessarily follow from the truth of the hypothesis S_1 and S_2 in view of Fig. 1 (a). Hence the argument is invalid.

(ii) The truth of the conclusion *S* that *x* is not an integer does not necessarily follow from the truth of the hypothesis S_1 and S_2 in view of Fig. 1(b). Hence the argument is invalid.

(iii) Here the truth of the statement S_2 that *x* is not a rational number is represented by placing a dot labelled *x* outside the set *Q* as shown in Fig. Since the dot *x* is outside the set *Q*, it is necessarily outside the set *Z* of all integers. Therefore, *x* is not an integer.



(c)

Thus, the truth of the conclusion *S* follows from the truth of the hypothesis S_1 and S_2. Hence the argument is valid.

8.5. EQUIVALENT IMPLICATION

If the two statements *p* and *q* are such that $p \Rightarrow q$ is true and the converse statement $q \Rightarrow p$ is also true, then $p \Rightarrow q$ are **equivalent** if and only if both $p \Rightarrow q$ and its converse $q \Rightarrow p$ both true.

For example, $3x - 2 = 10 \Rightarrow x = 4$ and $x = 4 \Rightarrow 3x - 2 = 10$ are both true.

Therefore, $3x - 2 = 10 \Rightarrow x = 4$.

For any two statements *p* and *q*, $p \Rightarrow q$ is equivalent to $\sim q \Rightarrow \sim p$ (but not $\sim p \Rightarrow \sim q$ or $q \Rightarrow p$).

The symbol '$\Leftrightarrow$' is the equivalent implication sign.

$p \Leftrightarrow q$ is read as '*p*' is equivalent to *q*'.

Note this carefully:

If $p \Rightarrow q$, then we can write the equivalent statement $\sim q \Rightarrow \sim p$.

8.6. VALID ARGUMENTS

An argument is **valid** if and only if the conclusion follows from other statement (the premises). The premise is the gives statement from which other conclusions can be drawn.

Note that the premise of an argument is always assumed to be true. Note also that the truth of the conclusion is irrelevent when testing for the validity of an argument. The fact that the conclusion is true is not sufficient for an argument to be valid.

Consider the following argument:

Monrovia is in Liberia

Liberia is in West Africa

Therefore Monrovia is in West Africa

The first two statements are the premises and the last statement is the conclusion

Note that although the two premises are false they are assumed to be true.

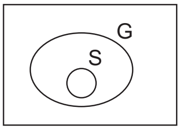
Note also that the conclusion is false although the argument is valid.

Example 3. Consider the statement:

q : Therefore is **no** soldier who does **not** use gun.

It means that *all* soldiers use gun. That is the set S = {soldiers} is a *subset* of the set G = {people who use gun}.

The statement can be represented in a Venn diagram as shown in figure.



S = G

Similarly, the statement:

"q : All policemen wear uniform" means

P = {people} is a *subset* of

U = {people who wear uniform}

Also, the statement:

"p : If students work hard **then** they will pass their examinations" means

S = {students who work hard} is a *subset* of {students who pass their examinations}

Example 4. *Consider the following statement:*

p : All soldiers are men

q : There is no sodier who does not use gun.

(*i*) *If M = {men}, S = {solidier) and G = {people with gun} draw a Venn diagram to illustrate p and q.*

(*ii*) *Statement whether or not each of the following is a valid conclusion from p and q.*

(*a*) *Men who use gun are soldiers*

(*b*) *All men use gun*

(*c*) *Men who do not use gun are not soldiers.*

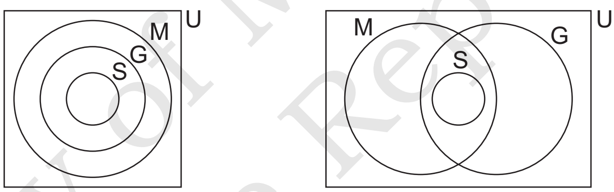
Solution. (*i*) Let U = {all people}, M = {Men}, S = {soldiers}

and G = {people with gun}

From stateents p and q we can write the following:

(1) $S \subset M$ and (2) $S \subset G$.

Figure (i) and (ii) are the possible diagrams illustrating the two statement p and q.



(i) (ii)

(ii) (a) Men who use gun are within region G but not all those who use gun are soldiers since a man can be inside region G but outside region S. Therefore the statement is not always true and not a valid deduction from p and q.

(b) All men are within region M but outside region G. Therefore the statements is not always true and not a valid deduction from p and q.

(c) Men who do not use gun are outside region G and are also outside region S. Therefore the statement is a valid deduction from p and q.

Example 5. *Consider the following statements.*

X : All soldiers are hardworking.

Y : No hardworking person is careless.

Draw a Venn diagram to illustrate the above statements.

Which of the following are valid conclusions from the statements X and Y?

(*i*) *Kwasi is a student* $\Rightarrow$ *Kwasi is not careless.*

(*ii*) *Asiedu is hardworking* $\Rightarrow$ *Asiedu is a student.*

(*iii*) *Efua is careless* $\Rightarrow$ *Efua is not a student.*

Solution. The statements are about students, hardworking persons, and careless persons, therefore;

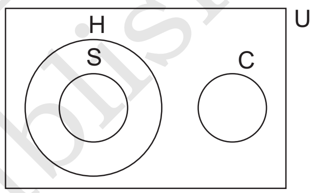
Let U = {all persons}, S = {hardworking persons} and

C = {Careless persons}

From the statements X and Y, we can write the following:

1. $S \subset H$ and H and C are disjoint sets.

Figure is the Venn diagram illustrating the statement X and Y.



(i) Kwasi is a student means he is in region S and therefore cannot be in region C. Therefore the statement

"Kwasi is a student $\Rightarrow$ Kwasi is not careless is valid.

(ii) Asiedu is hardworking means he is in region H and therefore can be within region S or outside S. Therefore the statement is not always true and not a valid conclusion.

(iii) Efua is careless means she is in region C and therefore cannot be in region S. Therefore the statement is always true and valid.

Using the fact that if $p \Rightarrow q$, then $\sim q \Rightarrow \sim p$.

We can also deduce the validity of an argument using the fact that; if $p \Rightarrow q$ is true then the equivalent statement $\sim q \Rightarrow \sim q$ is also true.

Example 6. *Consider the following statements.*

p : Abena has measles

q : Abena is in the hospital

If p $\Rightarrow$ q, state whether or not the following statement are valid.

(*i*) *If Abena is in the hospital, then she has measles.*

(*ii*) *If Abena is not in the hospital, then she does not have measles.*

(*iii*) *If Abena does not have measles, then she is not in the hospital.*

Solution. $p \Rightarrow q$ means q is true only if p is true.

We can also write the equivalent statement $\sim q \Rightarrow \sim p$

where $\sim p$ and $\sim q$ are the negations of the statements p and q respectively.

Note: We cannot write $q \Rightarrow p$ and $\sim p \Rightarrow \sim q$

(i) If Abena is in the hospital, then she has measles means $q \Rightarrow p$. Therefore the statement is not valid.

(ii) If Abena is not in the hospital, then she does not have measles means $\sim q \Rightarrow \sim p$. Therefore the statement is valid.

(iii) If Abena does not have measles, then she is not in the hospital means $\sim p \Rightarrow \sim q$. Therefore the statement is not valid.

Note that if $p \Rightarrow q$ we can write the equivalent statement $\sim p \Rightarrow \sim q$. But not $\sim p \Rightarrow \sim q$ or $q \Rightarrow p$.

Using the chain rule

The chain rule states that:

If p, q and r are any three statements

such that: $p \Rightarrow q$ and $q \Rightarrow r$, then $p \Rightarrow r$

Example 7. *Determine whether or not the following argument is valid:*

Monrovia is Liberia

Liberia is in West Africa

Therefore Monrovia is in West Africa

Solution. Let p : {Y is in Monrovia }

q : {Y is in Liberia} and

r : {Y is in West Africa}

The first premise means $p \Rightarrow q$ and the second premise means $q \Rightarrow r$

Hence by the chain rule $p \Rightarrow r$ *i.e.*, Monrovia is in West Africa

Therefore the conclusion follows from the premises and the argument is valid.

EXERCISE

1. Sentences involving variable time such as 'today' 'tomorrow' or 'yesterday' are not statements. "Tomorrow is Wednesday".
2. Sentences involving variable places such as 'here' 'there' are also not statements. "London is far from here".
3. Sentences involving pronouns time such as 'he' 'she' 'they' are not statements. 'He is a doctor'.
4. Use Venn diagrams to examine the validity of the following argument:

   S_1: If a man is a bachelor, he is unhappy.

   S_2: If a man is unhappy, he dies young.

   S_3: If All bachelors die young.

5. Consider the following statement:

X: *All* junior secondary pupils wear unfirom.

Y: *Most* junior secondary pupils are well behaved.

(*a*) Draw a venn diagram to illustrate the above statement.

(*b*) Using the venn diagram or otherwise determine which of the following implications are valid deductions from X and Y.

(*i*) Osei wears uniform $\Rightarrow$ Osei is a junior secondary pupil

(*ii*) Kofi is junior secondary pupils $\Rightarrow$ he is well behaved

(*iii*) Kwasi does not wear uniform $\Rightarrow$ he is not a junior secondary pupil

6. The following statements are true of a certain society:

p: *All* friends are intelligent

q: *No* intelligent person is conservative

(*a*) Draw a venn diagram to illustrate the above statement.

(*b*) Using your Venn diagram, complete the following statements

(*i*) Kweku who is not my friend ______

(*ii*) Ama who is not conservative ______

(*iii*) John who is intelligent ______

7. Consider the following statements:

X: *All* students with measles stay in the sick bay.

Y: *All* students in the sick bay *do not* do homework.

Which of the following statements is/are valid conclusion from the statements X and Y.

(*i*) Kofi does not have measles so kofi does his homework.

(*ii*) George has done his homework therefore he does not stay in the sick bay.

(*iii*) Jane does not have measles so she does not stay in the sick bay

8. Consider the following statements:

p: Kweku trains hard

q: Kweku wins the race

If $p \Rightarrow q$ which of the following statements are valid?

(*i*) If Kweku wins the race then he has trained hard

(*ii*) If Kweku does not train hard then he will not win the race

(*iii*) If Kweku does not win the race then he has not trained hard.

9. Consider the following statements:

p: Kwesi trains hard

q: Kwesi is rich

If $p \Rightarrow q$ which of the following statements are valid?

(*i*) If Kwesi is rich then works hard.

(*ii*) If Kwesi does not work hard the he is not rich.

(*iii*) If Kwesi does not work hard then he will not be rich.

PERIOD–III

M12CH9

TOPIC 9

Percentages

9.1. TAXES

The government of a country has responsibilities to provide some basic facilities and services to the people. These are water, electricity, roads, transportation and many others. For that, government needs money. Therefore, it deducts some percentage of amount from the salary or wages earned by the people of the country. This percentage of amount deducted is called tax. When there are more than one kind of tax, we call taxes.

The government of the country uses this money gained through taxes for the development of the country.

Taxes are of two types:

- Direct taxes
- Indirect taxes

Direct tax is levied directly on income of a person or an organisation. This tax, however, is not levied on the entire income of an individual. It is levied on a certain percentage of income called the 'taxable income'. The remaining percentage of income is called 'tax-free income'.

Indirect tax is the tax levied on goods and services.

Example 1. *Mrs. Esther's income is 200,000 L$ per month. Out of this amount 120,000 L$ is given to her as tax-free income. Calculate the amount of tax she will pay if she is taxed at the rate of 10% on the taxable income.*

Solution. Total income = 200,000 L$

Tax-free income = 120,000 L$

Therefore, taxable income = 200,000 L$ – 120,000 L$

= 80,000 L$

$$\text{Tax} = 10\% \text{ of } 80{,}000 \text{ L\$}$$
$$= \frac{10}{100} \times 80{,}000 \text{ L\$} = 8{,}000 \text{ L\$}$$

9.2. FINANCIAL PARTNERSHIP

A relationship between two or more **persons** who have agreed for setting up a business is known as **financial partnership.** The **persons** here are known as **financial partners**. There are two types of financial partners.

1. Active Partner 2. Silent Partner

An **active partner** is one who devotes his time for the business in addition to his investment.

A **silent partner** is one who merely invests money in the business.

A **'Firm'** is a name of the business under which all financial partners conduct their business activities.

The total money invested in a business is known as the **capital** of the business.

Basic Elements of Financial Partnership

There are mainly three basic elements of a financial partnership, which are explained below:

1. **Agreement:** It is a verbal or written commitment on account of which financial partnership comes into existence.
2. **Profit-sharing ratio:** An agreement among the financial partners to share profits or losses of the business.
3. **Principal-agent relationship:** This is also an agreement which explains the duties and rights of each financial partner.

Partnership Deed

An agreement between the financial partners of a firm is known as partnership deed or **partnership agreement**.

A partnership deed provides the ratio in which the financial partners share their profits or losses.

I. Calculating shares when capitals are different but period is same. The following working rule explains the whole procedure.

(i) Write the investments of the financial partners.

(ii) Calculate profit-sharing ratio (ratio of the capitals of the financial partners). If there are n financial partners, and let $r_1 : r_2 : r_3 : r_n$ be their profit-sharing ratios. Let the total profit be L$ P. Then, calculate the profit of each financial partner by using the following formulae.

Profit of 1st partner = L$ $\dfrac{r_1 P}{r_1 + r_2 + \cdots + r_n}$

Profit of 2nd partner = L$ $\dfrac{r_2 P}{r_1 + r_2 + \cdots + r_n}$ and so on.

Example 2. *Three financial partners Terry, Thomas and Noel start a business and invest the money such that the investment of Terry is equal to four times the capital of Thomas and capital of Thomas is 6 times the capital of Noel. Find the share of each financial partner if the total profit is L$ 478000.*

Solution. Let the capital of *Noel* = L$ x

∴ Capital of *Thomas* = L$ $6x$

Capital of *Terry* = L$ $(4 \times 6x)$ = L$ $24x$

The ratios of investments of *Terry*, *Thomas* and *Noel* = $24x : 6x : x$

∴ Profit-sharing ratio = 24 : 6 : 1

Total profit = L$ 478000

∴ Terry's share of profit = $\dfrac{(24) \times 478000}{24 + 6 + 1}$

$= \dfrac{11472000}{31} = 370064.51$

≅ L$ 3700065

Thomas' share of profit = $\dfrac{6 \times 478000}{34 + 6 + 1} = \dfrac{2868000}{31} = 92516.12$

≅ L$ 92516

Noel's share of profit = $\dfrac{1 \times 478000}{24 + 6 + 1} = \dfrac{478000}{31}$

= 15419.35 ≅ L$ 15419

I. Calculating shares when capitals are different and period is also different. We adopt the following procedure.

(i) Let there are n financial partners A, B, C, ... who invested L$ A_1, A_2, A_3, ... for the months T_1, T_2, T_3, ... respectively.

(ii) Calculate the adjusted effective capitals of each partner A, B, C, ...

The adjusted effective capital of A for one month

$$= A_1 \times T_1 = C_1, \text{ say,}$$

The adjusted effective capital of B for one month

$$= A_2 \times T_2 = C_2, \text{ say, and so on.}$$

(iii) Calculate profit-sharing ratios = $C_1 : C_2 : C_3$.

(iv) If the total profit is P, Then,

$$A\text{'s share of profit} = \text{L\$ } \frac{C_1 P}{C_1 + C_2 + \cdots + C_n}$$

$$B\text{'s share of profit} = \text{L\$ } \frac{C_2 P}{C_1 + C_2 + \cdots + C_n} \text{ and so on}$$

Example 3. *Esther, Annie and Fatu started their each own business and invested L$ 25000 each and for 6, 8 and 12 months respectively.*

Find the share of profit of each partner if the total profit earned is L$ 42625 after one year.

Solution. Here,
T_1 = 6 months, A_1 = L$ 25000
T_2 = 8 months, A_2 = L$ 25000
T_3 = 12 months, A_3 = L$ 25000

The adjusted effective capitals of Esther, Annie and Fatu are as under.

$$C_1 = A_1 \times T_1 = \text{L\$ } 25000 \times 6 = \text{L\$ } 1{,}50{,}000$$
$$C_2 = A_2 \times T_2 = \text{L\$ } 25000 \times 8 = \text{L\$ } 200000$$
$$C_3 = A_3 \times T_3 = \text{L\$ } 25000 \times 12 = \text{L\$ } 300000$$

$\therefore$ Profit-sharing ratios = $C_1 : C_2 : C_3 = 150000 : 200000 : 300000$
$= 15 : 20 : 30 = 3 : 4 : 6$

Here, total profit is P = L$ 42625

$$\therefore \text{ Esther share of profit} = \frac{3 \times 42625}{3+4+6}$$

$$= \frac{127875}{13} = 9836.53 \cong \text{L\$ } 9837$$

$$\text{Annie share of profit} = \frac{4 \times 42625}{3+4+6} = 13115.38 \cong \text{L\$ } 13115$$

$$\text{Fatu share of profit} = \frac{6 \times 42625}{3+4+6}$$

$$= 19673.076 \cong \text{L\$ } 19673.$$

9.3. INTEREST (PROFIT) ON CAPITAL

A partnership deed is an agreement between the partners of a firm. This agreement explains the duties and rights of each partner and in addition, it provides the following information.

(i) If states the proportion in which the profits of the business are to be shared.

(ii) It states the salary, if any, of each partner.

Further, if the partnership deed provides interest on capitals, then, the interest is given out of the total profit of the company before it is distributed among all the partners. The remaining profit is then distributed among the partners in the agreed proportions.

Example 4. *Prince and Joseph entered into a partnership investing L$12000 and L$ 15000 respectively. The partnership agreement provides for 3% interest on capitals.*

Find the share of each partner if the total profit earned is L$ 2700 after one year.

Solution. Since 3% interest is to be given on capitals invested by the partners, therefore,

$$\text{Prince interest} = 3\% \text{ of } 12000 = \frac{3}{100} \times 12000 = \text{L\$ } 360$$

$$\text{Joseph interest} = 3\% \text{ of } 15000 = 450$$

$\therefore$ Total interest to be paid to the partners = 360 + 450 = L$ 810

Also, total profit = L$ 2700

$\therefore$ Profit after distributing interest to Prince and Joseph

$$= 2700 - 810 = \text{L\$ } 1890$$

Hence, L$ 1890 is to be distributed among the partners in the ratio of their capitals. Now,

Ratio of the capitals = 12000 : 15000 = 4 : 5

$\therefore$ Prince share of profit $= \frac{4}{4+5} \times 1890 =$ L\$ 840

Joseph share of profit $= \frac{5}{4+5} \times 1890 =$ L\$ 1050

Further, Prince got = 840 + 360 = L\$ 1200

Joseph got = 1050 + 450 = L\$ 1500

Example 2. *Three partners Joseph, Peter and Musu start a new business and invest L\$ 10000, L\$, 15000 and L\$ 20000 respectively as a capital. Joseph receives 15% of the profit as an active partner and Peter gets 10% as manager after which the remaining profits are divided in proportion of capitals invested by each. Find the shares of profit of Peter and Musu if it is given that Hector receives L\$ 2394 as a profit.*

Solution. Let the total profit = L\$ x

Joseph profit as an active partner = 15% of L\$ x = L\$ $\frac{15x}{100}$ = L\$ $\frac{3x}{20}$

Peter profit as a manager = 10% of L\$ x = L\$ $\frac{x}{10}$

Extra profit of Joseph and Peter $= \frac{15x}{100} + \frac{x}{10} = \frac{25x}{100}$ = L\$ $\frac{x}{4}$

$\therefore$ Remaining profit = L\$ $\left(x - \frac{x}{4}\right)$ = L\$ $\frac{3x}{4}$

Hence, it is required to divide L\$ $\frac{3x}{4}$ among Joseph, Peter and Musu in the ratio of their capitals invested.

Ratio of invested capitals = 10000 : 15000 : 20000 = 2 : 3 : 4

$\therefore$ Joseph share of profit from the remaining profit

$$= \frac{2}{2+3+4} \times \frac{3x}{4} = \text{L\$ } \frac{x}{6}$$

Peter share of profit from the remaining profit

$$= \frac{3}{2+3+4} \times \frac{3x}{4} = \text{L\$ } \frac{x}{4}$$

Musu share of profit from the remaining profit

$$= \frac{4}{2+3+4} \times \frac{3x}{4} = \text{L\$ } \frac{x}{3}$$

Total share of profit received by Joseph

$$= \text{L\$}\left(\frac{x}{6}+\frac{3x}{20}\right) = \text{L\$}\ \frac{19x}{60} = \text{L\$}\ 2394 \qquad |\ \text{given}$$

$\Rightarrow \qquad 19x = 2394 \times 60 = 143640$

$\Rightarrow \qquad x = \frac{143640}{19} = \text{L\$}\ 7560$

$\therefore \qquad$ Total profit = L$ 7560

Total share of profit received by Peter

$$= \text{L\$}\left(\frac{x}{4}+\frac{x}{10}\right) = \text{L\$}\ \frac{7x}{20}$$

$$= \text{L\$}\ \frac{7}{20}\times 7560 = \text{L\$}\ 2646$$

Total share of profit received by Musu

$$= \text{L\$}\ \frac{x}{3} = \text{L\$}\ \frac{7560}{3} = \text{L\$}\ 2520$$

9.4. BANKING

Banking plays an important role for the Economical growth of a country and it is, in fact, the remote control of the money market in the country.

A bank may be defined as a **financial institution** where money and other valuables are deposited for safe keeping. Individuals and organizations that keep their money at the bank are called **customers.** A bank accepts money from its customers in the form of deposits which are usually repayable on demand or after a fixed period. Banks give safety to the deposits of its customers.

Transactions and Services Provided by a Bank

The basic transactions and services provided by a bank are:

I. Handling Different Types of Deposits

The bank collects deposits from the public. These deposits can be of different types as explained below:

(i) Savings deposit: This type of deposit encourages saving habit among the public. Excess money that will not be needed immediately is deposited into the account. The banks lend savings deposits to other customers as loans or overdrafts and charge interest on them. When the loan is paid back, the interest becomes profit for the banks. The bank then pays some of the profit as interest to the owner of the savings account.

Customers are allowed to withdraw part of their savings for use. *It is important that regular deposits are made into a savings account to keep the account active*. There is a minimum amount of money that must always be reserved in the account. This is known as the *minimum balance*. Savings account is suitable to salary and wage earners. It can also be opened in a single name by an individual or in joint names by groups and organizations.

(ii) Current deposit: This type of account is operated by businessmen. People who operate current account can withdraw money from their account more than once daily. The bank issues cheque books to current account holders which they fill and sign whenever they want to make withdrawals.

The banks are not allowed to lend deposits from current accounts as loans. No interest is paid on this account. Therefore, unlike savings deposits, banks do not earn interest or profits from current deposits. Thus, they also do not pay any interest on the current deposits to the customers.

(iii) Fixed deposit: In this type of account, a huge amount of money is deposited at one time for a fixed term. Higher interest is paid on fixed deposits, which varies with the period of deposit. The owner of the account will not be allowed to withdraw any part of the deposit before the fixed period ends.

(iv) Recurring deposit: This type of account is operated by salaried persons and petty traders. A certain sum of money is deposited into the bank from time to time. The account holder is permitted to make withdrawals, only after the end of certain period. A higher rate of interest is paid on the deposits to the account holder.

II. Granting Loans and Advances

Banks advance loans to their customers, the business community and other members of the public. The rate charged is higher than what it pays on deposits. The difference in the interest rates is its profit. The common types of bank loans and advances are overdrafts and loans.

(i) Overdraft: This type of advance is given to current account holders. This means that they can withdraw more than they have in their account. A certain amount is sanctioned as overdraft which can be withdrawn within a certain period of time say three months or more. Interest is charged on actual amount withdrawn. For example, if the balance in a certain current account is L$ 1800 and the account holder drew a cheque of L$ 2000 and it is honoured, it means he has overdrawn the account in excess of L$ 200.

Loan: It is normally for a fixed term say a period of one year or a period of five years. Since recent times, banks also lend money for long term. Repayment of money can be in the form of installments spread over a period of time or in a lump sum amount.

Interest is charged on the actual amount sanctioned, whether withdrawn or not. The rate of interest may be slightly lower than what is charged on overdrafts and cash credits. Loans are normally secured against tangible assets of the person or company contracting the loan.

III. Issuing Bank Drafts and Demand Draft

A bank draft is a document issued by one bank against funds deposited into its account at another bank, authorizing the second bank to make payments to the individual named in the draft.

A **demand draft** is also a bank document but used by individuals to make/transfer payments from one bank account to another. A demand draft requires no singnature in order to be cashed.

IV. Payment Cheque

It is a bank document that orders a bank to pay a specific amount of money from a person's account to the person whose name is mentioned on the document. A specimen copy of a payment cheque of Bank of Liberia and Prudential Bank Limited.

Ad Valorem duty: It is the import duty on goods. Let C.I.F. or F.O.B value is x, then, Ad Valorem duty = x% of C.I.F. or x% of F.O.B.

Example 6. *A car was freight in United Kingdom for $ 2206.98. The freight charges were $ 325.86 and the insurance cost $ 25. Find the C.I.F. value of the car. If the import duty on cars is 25% Ad valorem, find the duty on the car.*

Solution. Here cost of the car = $ 2206.98

Insurance charges = $ 25

Freight charges = $ 325.86

$\therefore$ C.I.F. Value = 2206.98 + 25 + 325.86 = \$ 2557.84

Also, duty on the car = 30% of C.I.F. value

$$= \frac{30}{100} \times 2557.84 = 767.352 \cong \$\ 767.$$

9.5. VALUE ADDED TAX (VAT)

The Value Added Tax (VAT) was introduced in 2013 by the provisions of **VAT act** (Act 870). This is an indirect tax and imposed on consumers when they purchase goods. The rate is 15% for businesses and individuals whose annual turnover is L\$ 120000 or above on the value of goods and services.

If the VAT rate on goods or services is r%, Then,

VAT = r% of VAT exclusive cost or

VAT = r% of Basic cost,

where basic cost is the cost of item without VAT. We also define

$$\text{VAT} = \frac{r}{100 + r} \times \text{VAT inclusive cost.}$$

National Health Insurance Levy (NHIL)

It is a levy imposed on goods and services supplied in or imported into the country. All goods and services are subject to the levy unless they are otherwise exempted. The levy is charged at a rate of $2\frac{1}{2}\%$. The NHIL is collected by the Domestic Tax Revenue Division governed by Ghana Revenue Authority.

Example 7. *George purchased a hair dryer for L\$ 230 including 15% VAT. Find the price of the hair dryer before VAT.*

Solution. Let, the price of hair dryer before VAT = L\$ x

$$15\% \text{ of } x = \frac{15}{100} \times x = \frac{3x}{20}$$

As per given, VAT inclusive cost = L\$ 230

$$\Rightarrow \quad x + \frac{3x}{20} = 230 \quad \Rightarrow \quad \frac{20x + 3x}{20} = 230$$

$$\Rightarrow \quad 23x = 230 \times 20 = 4600$$

$$\Rightarrow \qquad x = \frac{4600}{23} = \text{L\$ } 200$$

∴ The price of hair dryer before VAT = L$ 200

Example 8. *Peter brought a personal computer costing L$ 1800 at L$ 1980 after paying the VAT. Find the rate at which the VAT was charged.*

Solution. VAT exclusive, price of the computer = L$ 1800. VAT inclusive price of the computer = L$ 1980

∴ VAT charged = 1980 – 1800 = L$ 180

Let r be the rate of VAT, then,

VAT = rate% × VAT exclusive price of the computer

$$\Rightarrow \qquad 180 = \frac{r}{100} \times 1800 \quad \Rightarrow \quad r = 10$$

∴ The required VAT rate = 10%.

Example 9. *The marked price of a coat and paint is L$ 980. If the VAT on the coat is 10% and that on the paint is 5% and if the total VAT is L$ 94, calculate the marked price of the coat and paint.*

Solution. Let the marked price of the coat = L$ x and the marked price of the paint = L$ y

As per given, $x + y = 980$ ...(1)

VAT on the coat = 10% of marked price of the coat

$$= \frac{10}{100} \times x = \frac{x}{10} \qquad \text{...(2)}$$

VAT on the paint = 5% of marked price of the paint

$$= \frac{5}{100} \times y = \frac{y}{20} \qquad \text{...(3)}$$

Adding (2) and (3),

$$\text{Total VAT} = \frac{x}{10} + \frac{y}{20} = 94 \qquad |\text{Given}$$

$$\Rightarrow \qquad 2x + y = 1880 \qquad \text{...(4)}$$

Subtracting (1) from (4), we get

$$x = 1880 - 980 = 900$$

Again, (1) gives $y = 980 - x = 980 - 900 = 80$

∴ Marked price of the coat = L$ 900

Marked price of the paint = L$ 80

9.6. HOUSE HOLD BILLS

I. Calculation of Electricity Bill

The consumption of electrical energy by homes and small business is usually measured in kilowatt hours. Thus, the

energy (E) = Power (P) × time (t).

If the unit of power P is watt, and of time sec, then, the unit of energy E is watt second.

If the unit of power P is kilowatt (kW) and of time is hour, then, the unit of energy E is kilowatt hour (kWh).

Thus, one kilowatt-hour is the amount of electrical energy consumed by a 1 kW device in one hour.

For example, suppose 1.5 kW electrical heater runs for 2 hours, then, the energy consumed by the heater can be calculated as below:

Here P = 1.5 kW, t = 2 hrs

∴ Energy (E) = (1.5 kW) × (2 hr) = 3 kWh

Example 10. *Annie spends half an hour each day drying his hair with an electric hair dryer with a power rating of 1.5 kW. The unit cost of electricity is GHp 21 per kWh. Find how much Annie spent on drying his hair each* (i) *week* (ii) *month?*

Solution. (i) Here P = 1.5 kW, t = 0.5 hour per day

= 7 × 0.5 hours per week

= 3.5 hours per week

∴ The energy used in one week is given as

$E = P \times t =$ 1.5 (kW) × 3.5 (hours)

= 5.25 kWh

One unit cost = GHp 21 per kWh

∴ Total cost for one week = 5.25 × 21

= GHp 110.25 = L$ 1.1025

(ii) Take one month = 30 days

∴ t = 0.5 hour per day

= 0.5 × 30 = 15 hour per month

The energy used in one month is given as

$E = P \times t =$ 1.5 (kW) × 15 hr = 22.5 kWh

∴ Total cost for one month

$= 22.5 \times 15 = \text{GHp } 337.5 = \text{L\$ } 3.375.$

II. Calculation of Water Bill

The consumption of water by an individual or group of individuals is measured in kilolitre (kL). The rates as approved by the pure as published in Gazette No. 31, dated 2nd April 2015 are given in the following Table.

Table. Water Tariffs

Category of Service Consumption	***Monthly (1000 litres)***	***Approved Rates in Ghp/1000 litres***
(a) Metered Domestic	0–20	178.3326
	21 and above	267.3313
(b) Commerical/Industrial	Flat Rate	380.0075
(c) Public Institutions/Govt. Departments	Flat Rate	342.9438
(d) Unmetered Premises-Flat rate per house per month		1160.7090
(e) Premises without connection (Public stand pipes per 1000 litres)		176.3036
(f) Special Commercial per 1000 litres		1080.6204

Example 11. *In Liberia, water is charged according to the quantity used. A Typical Terrif in given below.*

Category	***Rates in GHp/1000 litres***
0 kL – 9 kL	*178*
10 kL – 25 kL	*267*
26 kL – 45 kL	*380*
45 kL and above	*392*
Service charge	*Nil*
VAT	*15%*

(a) *Gyamfua's monthly water consump-tion is 32000 litres. Calculate his monthly water bill including VAT @ 15%.*

(b) *Find the amount he paid per kilolitre.*

Solution. (a) Break 32000 litres = 32 kL into the different categories as listed in the given table.

In 0 kL – 9 kL category, the water he used = 9 kL

∴ The amount he paid = 178 × 9 = GHp 1602 = L$ 16.02

In 10 kL – 25 kL, category the water he used

= 16 kL (9 + 16 = 25)

∴ The amount he paid = 267 × 16 = GHp 4272 =L$ 42.72

In 26 kL – 45 kL, category the water he used = 7 kL (25 + 7 = 32)

∴ The amount he paid = 380 × 7 = GHp 2660 = L$ 26.60

Total amount he paid = 16.02 + 42.72 + 26.60 = L$ 85.34

Also, VAT @ 15% of 85.34 = $\frac{15}{100} \times 85.34$ = L$ 12.801

Hence, the total amount paid by him

= 85.34 + 12.801 = L$ 98.141

(b) He paid L$ 98.141 for 32 kL

∴ for 1 kL, he will pay = $\frac{98.141}{32}$ = L$ 3.07

III. Calculation of Telephone Bill

Example 12. *Mobile Telecommunication Network, Accra, Liberia, sends a bill for the services used by a consumer.*

Name: Ms. Emelia	***Peak hours***	***off-Peak hours***	***Cost (in L$)***
Local calls (in minutes)	*190*	*250*	*0.11 per minute*
Data (in MB)	*12*	*15*	*0.2 for first 3 MB* *0.15 for next 3 MB* *0.1 for next 4 MB and above*
STD calls (in minutes)	–	*8*	*1.10 per minute*

Assume that there is a standing charges of L$ 18.50 and the off-Peak minutes are sold at 50% discount.

Find the amount of the Telephone bill which Emelia has to pay.

Solution. (i) Local calls charges (Peak)

$$= 190 \times 0.11 = \text{L\$ } 20.9 \quad \ldots(1)$$

Local calls charges (off-Peak)

$$= 250 \times \left(0.11 \times \frac{50}{100}\right) = 250 \times \left(\frac{0.11}{2}\right)$$

$$= \text{L\$ } 13.75 \quad \ldots(2)$$

(ii) Data charges (Peak)

First 3 MB = 3 × 0.2 = L$ 0.6 ...(3)

Next 3 MB = 3 × 0.15 = L$ 0.45 ...(4)

Remaining 6 MB = 6 × 0.1 = L$ 0.6 ...(5)

Data charges (off-Peak)

$$\text{First 3 MB} = 3 \times \left(\frac{0.2}{2}\right) = \text{L\$ } 0.3 \quad \ldots(6)$$

$$\text{Next 3 MB} = 3 \times \left(\frac{0.15}{2}\right) = \text{L\$ } 0.225 \quad \ldots(7)$$

$$\text{9 MB} = 9 \times \left(\frac{0.1}{2}\right) = \text{L\$ } 0.45 \quad \ldots(8)$$

(iii) STD calls charges (Peak) = Nil

$$\text{STD calls charges (off-Peak)} = 8 \times \left(\frac{1.10}{2}\right) = \text{L\$ } 4.4 \quad \ldots(9)$$

(iv) Standing charges = L$ 18.50 ...(10)

Adding (1), (2), ... (10), we have

∴ The total amount of Emelia's bill = L$ 60.175

9.7. HIRE PURCHASE

The term **hire purchase** means a way of buying expensive goods such as T.V., Friz, Car, Flat etc. Using hire purchase, one pays a small amount in the beginning and the remaining amount to be paid by monthly or yearly installments.

Note: Buying of goods on hire purchase usually costs more to calculate the total hire purchase price, we add the deposit and the total of all of the installments.

Example 13. *A pressure cooker can be bought on hire purchase price by paying a deposit of 35% and 36 monthly payments of L$ 11.40. The cost of pressure cooker in cash price is L$ 460. Find the hire purchase price of the pressure cooker.*

Solution. Deposit amount = 35% of L$ 460 = $\frac{35}{100} \times 460$ = L$ 161

One installment cost = L$ 11.40

$\Rightarrow$ 36 installments cost = 36 × 11.40 = L$ 410.4

$\therefore$ Hire purchase price

= Deposit amount + Total installments cost

= 161 + 410.4 = L$ 571.40

Example 14. *Electroland Liberia Limited (EGL) launched a new T.V. which costs L$ 2,190 on cash payment. It is available on hire purchase price by paying a deposit of 15% followed by 12 installments of L$ 178.50. Find the total hire purchase price and the extra money that you would pay (over the cash price) using hire purchase.*

Solution. Deposit amount = 15% of L$ 2,190 = $\frac{15}{100} \times 2190$ = L$ 328.5

1 installment cost = L$ 178.50

$\therefore$ 12 installments cost = 178.50 × 12 = L$ 2,142

$\therefore$ Hire purchase price

= deposit amount + 12 installments cost

= 328.5 + 2142 = L$ 2,470.5

Given, cash price = L$ 2,190

$\therefore$ Extra amount to be paid

= Hire purchase price – Cash price

= 2470.5 – 2190 = L$ 280.5

Example 3. *The cash price of a computer in Accra is $ 550. The hire purchase price is $ 625. If you pay a deposit of 15% followed by 20 equal monthly installments, find how much you pay (in L$) per month. The rate of exchange is L$1 = $ 0.257732*

Solution. Deposit amount = 15 % of $ 550 = $\frac{15}{100} \times 550$ = $ 82.50

$\therefore$ 20 installments cost

= Hire purchase price – Deposit amount

= 625 – 82.50 = \$ 542.50

$\Rightarrow$ 1 installment cost = $\frac{542.50}{20}$ = L\$ 27.125

We need to find the cost of one installment in Liberia currency.

Given L\$ 1 = \$ 0.257732

$\Rightarrow$ \$ 1 = L\$ $\frac{1}{0.257732}$ = L\$ 3.879999379

$\therefore$ \$ 27.125 = 3.879999379 × 27.125

= L\$ 105.2449832

Thus, one has to pay = L\$ 105.25 per month.

EXERCISE

1. Mr. James income is 2500,000 L\$ per month. The tax-free income is 150,000 L\$. He is taxed at the rate of 10% for the first 40,000 L\$ 15% for the next 40,000 L\$ and 20% on the remaining. Calculate the total amount of tax he will pay.
2. Liberia Electronics limited is Governed by three partners James, Prince and Joseph. James receives $\frac{2}{3}$ of the total capital. Prince and Joseph divide the remainder equally. Frank's income is increased by L\$ 500 when the rate of profit rises from 5% to 7%. Find the capital of Prince and Joseph.
3. Katumi, Kwame and Napolean form a partnership to start a new business. Their shares are in the proportions $\frac{1}{3}:\frac{1}{4}:\frac{1}{5}$. Katumi withdraws half his capital at the end of 15 months and after 15 months more, a profit of L\$ 6580 is divided. Find the share of each partner.
4. Sarah, Seidu and Benjamin created a new firm by investing capitals of L\$ 60000, L\$ 70000 and L\$ 85000 respectively. The partnership deed provides for 5% interest on capitals, an annual salary of L\$ 10000 to Sarah and travelling allowance of L\$ 3000 to Benjamin before distributing profit of the firm. If the profit earned

by the firm is L$ 60000 and the partners agree to share the profits in their capital ratios, find the share of profit of each partner.

5. Joseph, Amadu, Alex and Jimmy entered into a partnership investing L$ 55000, L$ 55000, L$ 45000 and L$ 40000 for 6, 7, 8 and 4 months respectively Alex is an active partner and gets 10% of the total profit. Find the share of each partner if the total profit earned after a year is L$25000.

6. Futa purchased an item for L$ 330 including 10% VAT and a mobile accessory for L$ 212 including 6% VAT. Find the marked price of the item and the mobile accessory.

7. The cost of a DVD player, inclusive of VAT and NHIL, is L$ 625.50. The VAT is charged at the rate of 15% and NHIL at 2.5% respectively. Find

(a) the cost of DVD player (VAT and NHIL exclusive)

(b) The NHIL charged

(c) The VAT charged

8. How long will the following devices run on 1 kWh of energy?

(a) Hair dryer (1.5 kW) (b) Television (150 W)

(c) Electric cooker (4 kW) (d) Vacuum cleaner (800 W)

9. A DVD player is priced at L$ 320 in different shops A and B which offer different hire purchase terms. Shop A requires 20% deposit and 12 monthly installments of L$ 26.60. Shop B requires 30% deposit and 12 monthly installments of L$ 23.50. Which shop has the better deal?

M12CH10

TOPIC 10

Rigid Motion-II & Enlargement

10.1. ROTATIONAL SYMMETRY

An object is said to have a **rotational symmetry** if it can be turned around its centre to match itself in less 360° turn.

Order (or degree) of rotational symmetry

The number of times an object will fit onto itself in one complete rotation is called the order (or degree) of the rotational symmetry.

Note: The order (or degree) of rotational symmetry of a regular polygon of n sides is n. We shall see it later.

Angle of rotational symmetry

The minimum angle required for an object to rotate, either in clockwise or anti-clockwise direction, and coincide with itself is known as angle of rotational symmetry and is given by

$$\frac{360°}{\text{Order (or degree) of rotation}}.$$

Example 1. *State the rotational symmetry of the following geometrical shapes and find the angle of rotational symmetry.*

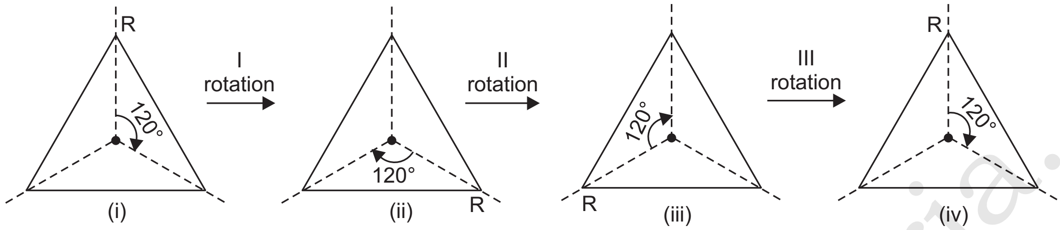
(a) *Equilateral triangle* (b) *Square*

(c) *Regular pentagon* (d) *Regular hexagon*

(e) *Rectangle*

Solution. (a) An equilateral triangle is a regular polygon of 3 sides, therefore, its order of rotational symmetry is 3 (see figure). Also,

$$\text{Angle of rotational symmetry} = \frac{360°}{3} = 120°$$

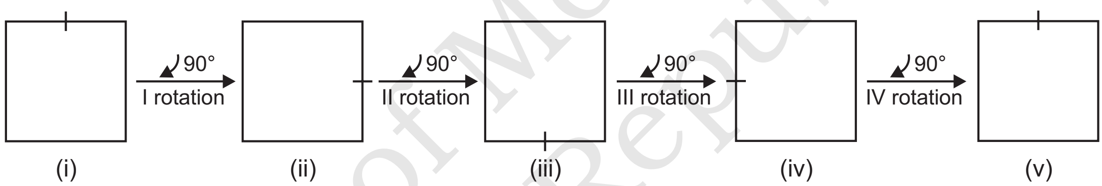


Order of rotational symmetry of an equilateral triangle

Thus, figure (i) has been rotated three times about an angle 120° in clockwise direction to get figure (iv) which is exactly same as figure (i). Hence, the order of rotational symmetry of an equilateral triangle is 3 and angle of rational symmetry is 120°.

(b) A square is also a regular polygon of 4 sides. Therefore, its order of rotational symmetry is 4 (see figure). Also,

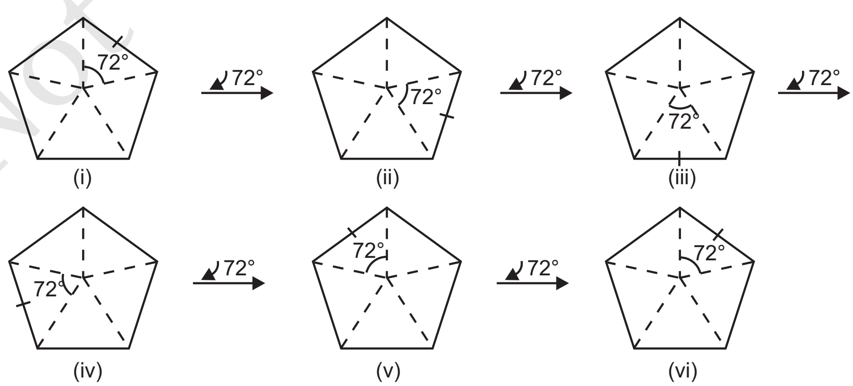
$$\text{Angle of rotational symmetry} = \frac{360^\circ}{4} = 90^\circ$$



Order of rotational symmetry of a square.

Thus, figure (i) has been rotated 4 times about an angle 90° in clockwise direction, to get figure (v) which is exactly same as figure (i). Hence, the order of rotational symmetry of a square is 4 and angle of rotational symmetry is 90°.

(c) The order of rotational symmetry for a regular pentagon (number of sides is 5) is 5 (see figure) and

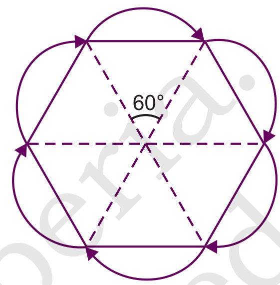


Order of rotational symmetry of a regular pentagon.

Angle of rotational symmetry = $\frac{360°}{5}$ = 72°.

(d) The order of rotational symmetry of a regular hexagon (number of sides is 6) is 6 (see figure). Also,

Angle of rotational symmetry = $\frac{360°}{6}$ = 60°.



Regular hexagon

(e) If a rectangle is rotated by 180° twice, it comes to its original position. (see figure)

∴ Order of rotational symmetry is 2.

Also, angle of rotational symmetry = $\frac{360°}{2}$ = 180°

Example 2. *Consider the following Ghananian coins and assume that any two vertices of coin (iii) is joined by a straight line as shown in Fig. For each coin.*

(i) (ii) (iii)

(a) *Write the geometrical name.*
(b) *Find the order of rotational symmetry.*
(c) *Find the angle of rotational symmetry.*

Solution.

(i) (a) The geometrical name is circle.
(b) Order of rotational symmetry is infinite.
(c) Angle of rotational symmetry = $\frac{360°}{\infty}$ = 0°.

(ii) (a) The geometrical name is regular heptagon.
(b) Order of rotational symmetry = 7
(c) Angle of rotational symmetry = $\frac{360°}{7}$ ≅ 51.42°.

(iii) The geometrical name is regular polygon of 12 sides.
Order of rotational symmetry = 12

Angle of rotational symmetry = $\frac{360°}{12}$ = 30°

10.2. ROTATION

A rotation is also a rigid motion that turns every point of a pre-image through a specified angle and direction about a fixed point.

The fixed point is called the **centre of rotation** and the amount of rotation (measured in degrees) is known as angle of rotation.

Rotation of Objects in Everyday Life

In our everyday life, we come across many objects, such as clock hands, ceiling fans, lid of a jar when the jar is opened or closed, tyres of a truck which rotate as shown in figure.

Clock hands


Ceiling fans


Lid of a jar


Wheels of a truck

When an object rotates, its shape and size do not change. Rotation takes place about a point. This point may or may not be the centre of the object.

Clockwise and anti-clockwise directions: Looking at the clock

The hands of the clock always rotate in a particular direction, which is known as **clockwise direction** and is denoted by the symbol ↻. There are many objects which can rotate in **anti-clockwise direction** (direction opposite to the hands of the clock) as well as in clockwise direction such as pinwheel, (see figure) lid of a jar, tyres of a truck etc. The anti-clockwise direction is also known as **counter-clockwise direction.**


Pinwheel

Meaning of full turn, half-turn and quarter-turn.

When an object is rotated through an angle of 360°, then, the rotation is known as **full turn rotation**.

When the angle of rotation is 180° (half of 360°), then, the rotation is known as **half turn rotation**. Similarly, a rotation through 90° ($\frac{1}{4}$ of 360°) is known as quarter rotation.

Note 2: A rotation requires centre of rotation, direction of rotation and angle of rotation.

Rotation of a point by 90° about the origin.

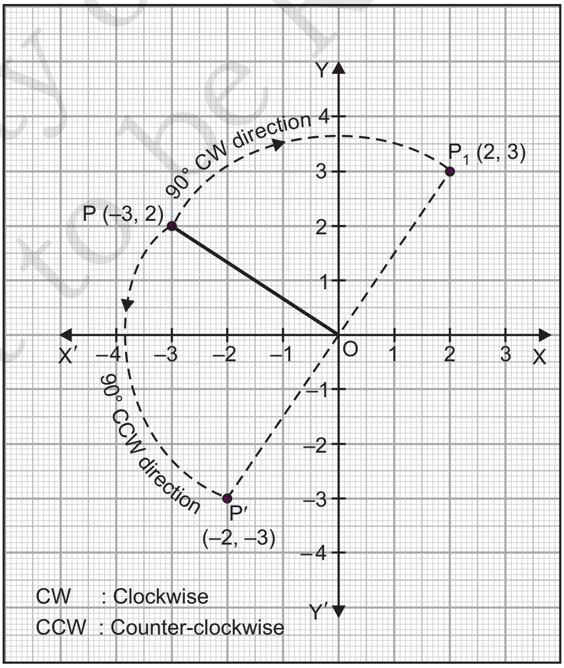
Rule: Suppose a point $P(x, y)$ is rotated through an angle of 90° **counter-clockwise** about the origin, then, the image of $P(x, y)$ after rotation is given by $P'(-y, x)$.

If the point $P(x, y)$ is rotated through an angle of 90° **clockwise** about the origin, then, the image of $P(x, y)$ after rotation is given by $P'(y, -x)$.

Example 3. *Suppose a point (–3, 2) is rotated through an angle of 90°. Find its images if the point is rotated* (i) *clockwise* (ii) *anti-clockwise. Draw the images on the graph or square paper.*

Solution. Let the point is $P(-3, 2)$.

When the point $P(-3, 2)$ is rotated by 90° anti-clockwise (or counter-clockwise) its image will be $P'(-2, -3)$.



Images of the point (–3, 2)

When the point $P(-3, 2)$ is rotated by 90° clockwise, its image will be $P_1(2, 3)$.

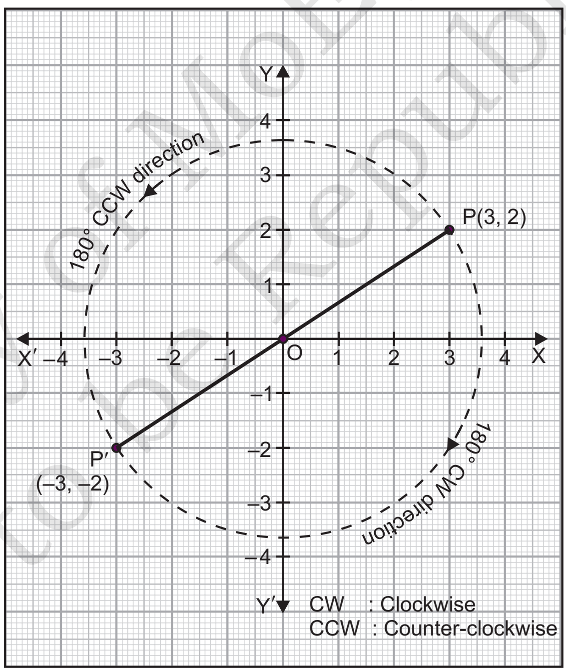
The images are shown on the graph paper (see figure).

Rotation of a point by 180° about the origin

Rule: Suppose a point $P(x, y)$ is rotated through an angle of 180° in clockwise or counter-clockwise direction, then, its image will be $P'(-x, -y)$. Thus, a rotation through 180° needs **no direction**.

Example 4. *Suppose a point P(–3, 2) is rotated through an angle of 180°, either clockwise or counter-clockwise. Find and draw its image point on the graph or square paper.*

Solution. A rotation through 180° needs no direction. The image of $P(3, 2)$ is the point $P'(-3, -2)$ and is shown in Fig.



Images of the point (3, 2)

Rotation of a point by 270° about the origin

Rule: Suppose a point $P(x, y)$ is rotated through an angle of 270° counter-clockwise, then, its image will be $P'(y, -x)$, which is same if the point $P(x, y)$ is rotated by 90° clockwise. Hence CCW rotation by 270° = CW rotation by 90°

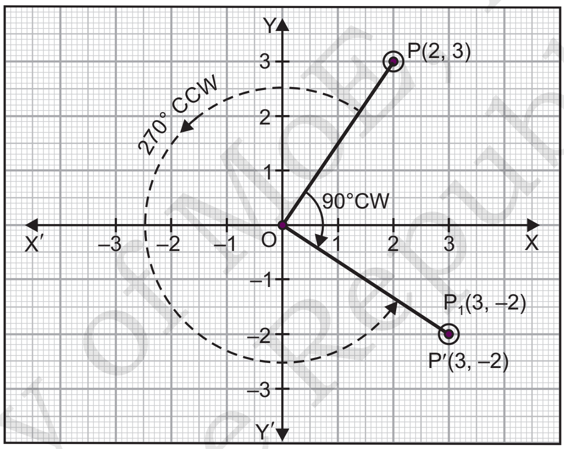
Similarly, if the point $P(x, y)$ is rotated through an angle of 270° clockwise, then, its image will be $P_1(-y, x)$, which is same if the point $P(x, y)$ is rotated by 90° counter-clockwise. Hence,

CW rotation through 270° = CCW rotation through 90°.

Example 5. *Find and draw the image of the point P(2, 3) if it is rotated by 270° counter-clockwise. Also, draw the image of P(2, 3) if it is rotated by 90° clockwise. Are the images same?*

Solution. If the point $P(2, 3)$ is rotated by 270° counter-clockwise, its image will be $P'(3, -2)$, which is shown on the graph paper (see figure).

If the point $P(2, 3)$ is rotated by 90° clockwise, its image will be $P_1(3, -2)$, which is same as $P'(3, -2)$ (figure).



Images of the point (2, 3)

Rotation of a point by 360° about the origin

Rule: Suppose a point $P(x, y)$ is rotated through an angle of 360° either in clockwise or counter-clockwise direction, then, the image of $P(x, y)$ will be the same point $P(x, y)$.

Rotation of plane figures about the origin

The procedure is explained in the following examples.

Example 6. *Graph the image of Fig. when it is rotated by*

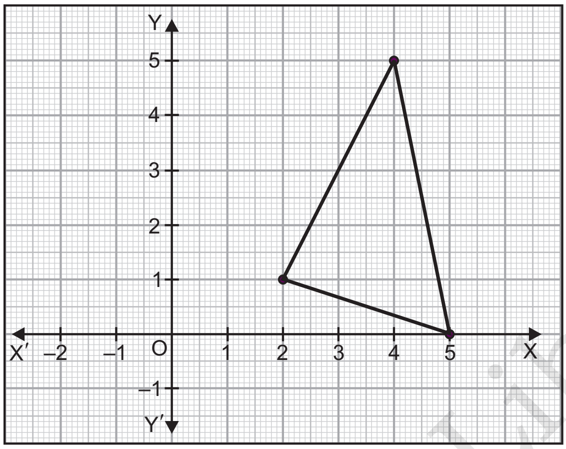
(a) *90° counter-clockwise about the origin.*

(b) *90° clockwise about the origin.*

(c) *270°clockwise about the origin.*

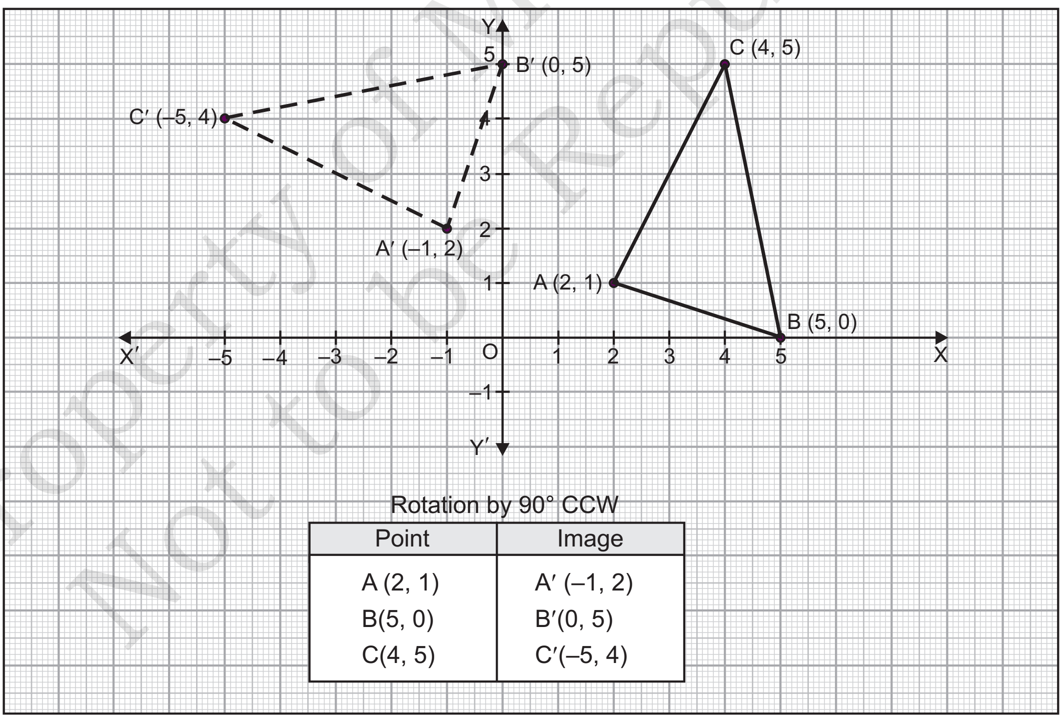
(d) *270° counter-clockwise about the origin.*

Also, write the vertices of the image triangle in each case.



Solution. Let the vertices of the given triangle are $A(2, 1)$, $B(5, 0)$ and $C(4, 5)$ respectively.

(a) If the triangle ABC is rotated by 90° counter-clockwise, then, the image triangle $A'B'C'$ is shown in the following graph (see figure).

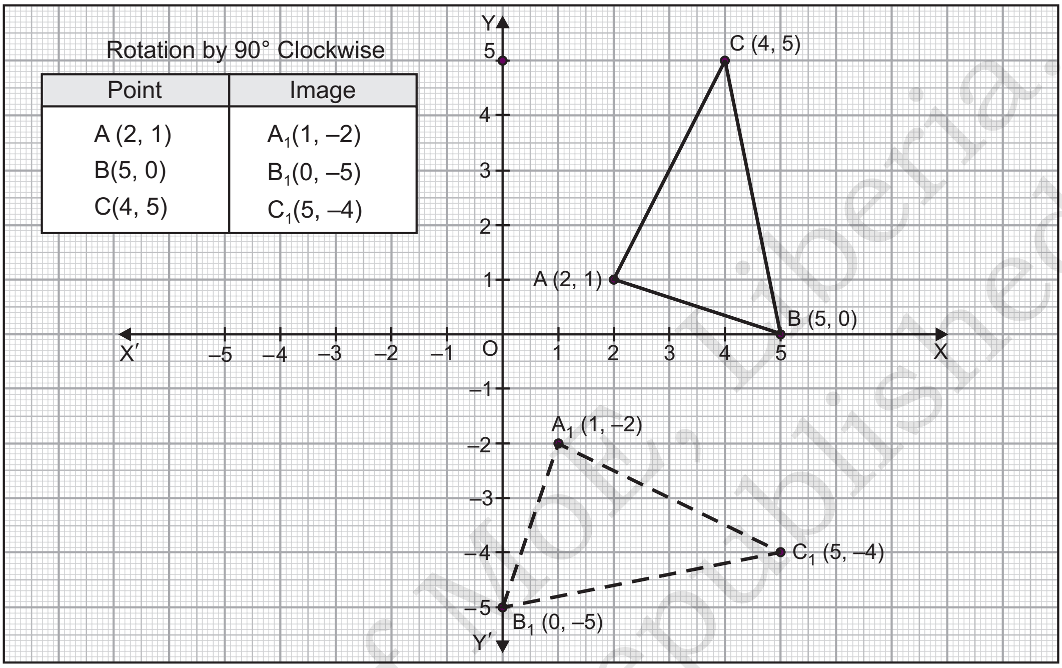


Rotation by 90° CCW

Point	Image
A (2, 1)	A′ (–1, 2)
B(5, 0)	B′(0, 5)
C(4, 5)	C′(–5, 4)

Image of triangle *ABC* after rotating by 90° counter-clockwise.

(b) If the triangle ABC is rotated by 90° clockwise then, the image triangle $A_1B_1C_1$ is shown in the following graph (see figure).



Rotation by 90° Clockwise

Point	Image
A (2, 1)	A_1(1, –2)
B(5, 0)	B_1(0, –5)
C(4, 5)	C_1(5, –4)

Image of triangle *ABC* after rotating by 90° clockwise

(c) Since, rotation through 270° clockwise

= rotation through 90° counter clockwise,

∴ The image triangle will be same as in part (a)

(d) Since rotation through 270° counter clockwise

= rotation through 90° clockwise,

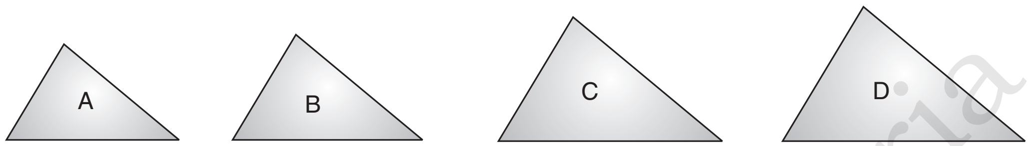
∴ The image triangle will be same as in part (b).

10.3. ENLARGEMENTS

In our previous discussion, we have learnt about three main transformations: translation, reflection and rotation. The other important transformation is enlargement.

Drawing enlargement of geometrical figures with given scale factors

Consider the following figure:



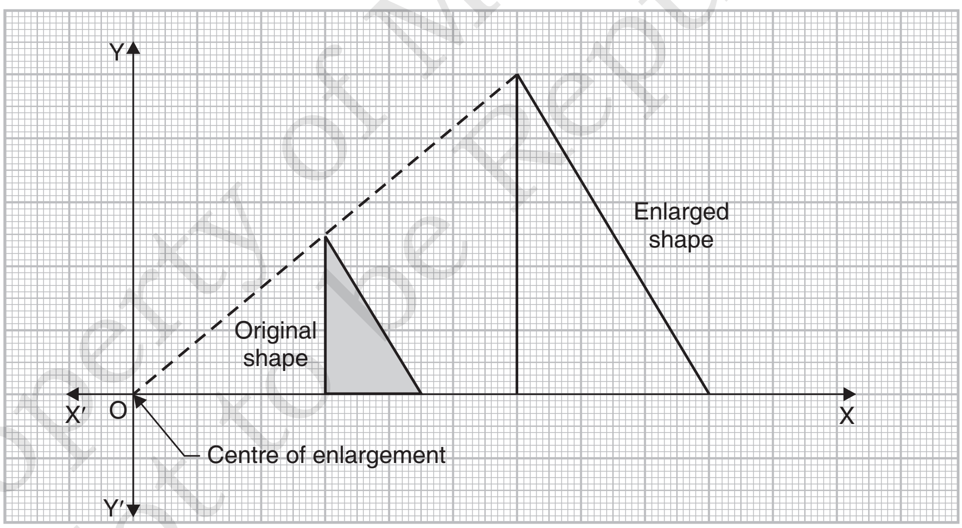
Shapes *B*, *C* and *D* are all enlargements of shape *A*. However, they are all in different positions and of different sizes.

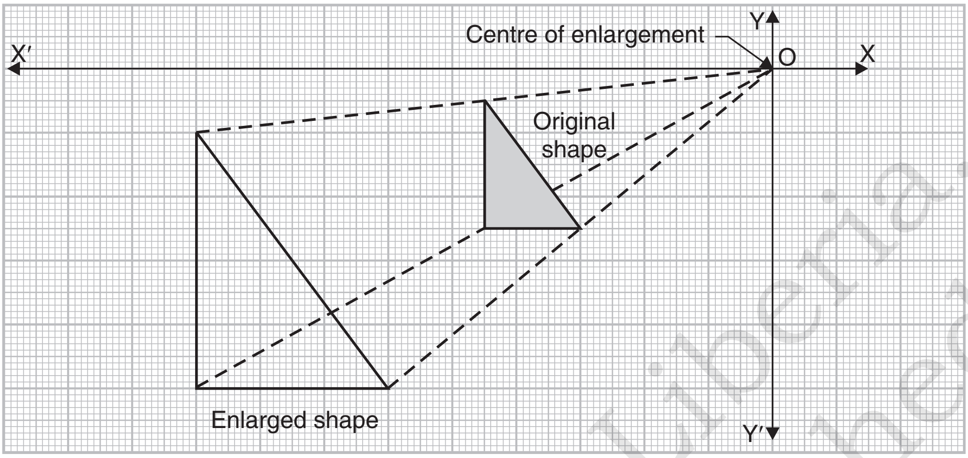
When drawing the enlargement of a geometrical figure, we must state their positions and their sizes.

Centre of Enlargement

The position of the enlarged shape (figure) is described by the centre of enlargement.

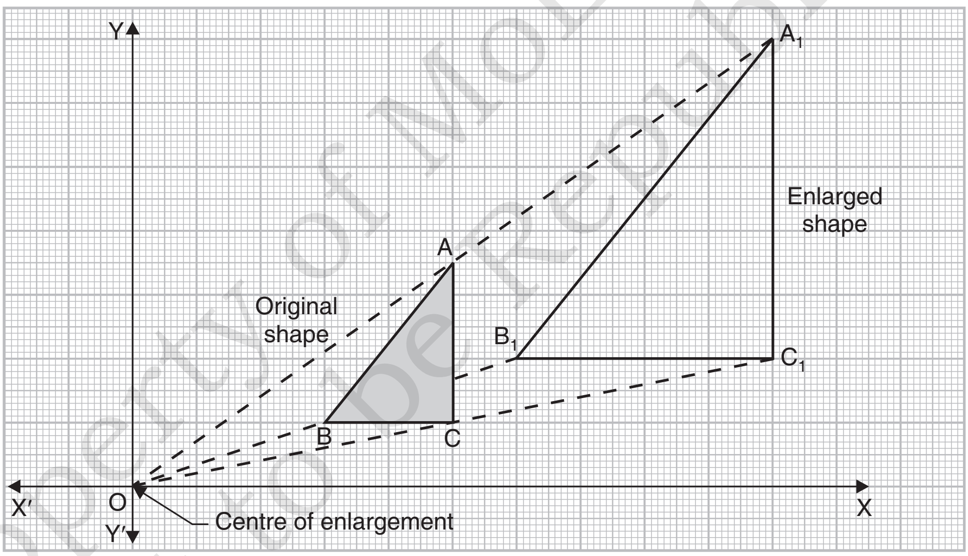
In the following enlargements, [see figure (a–b)] point *O* is the centre of enlargement.





Scale factor

The size of an enlargement is described by its scale factor.



For example, a scale factor of 2 means that the new shape (figure) is twice the size of the original. A scale factor of 3 means that the new shape is three times the size of the original. Triangle $A_1B_1C_1$ (see figure) is an enlargement of triangle ABC with a scale factor of 2.

What do you observe?

For a scale factor of 2,

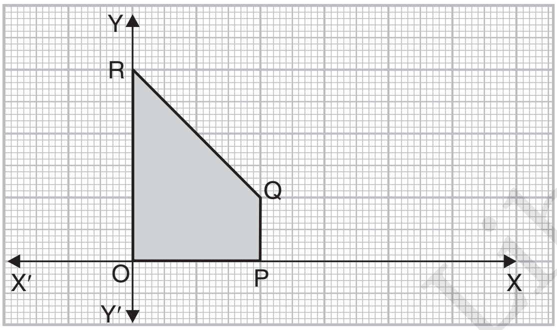
$OA_1 = 2 \times OA$, $OB_1 = 2 \times OB$, $OC_1 = 2 \times OC$

Points A_1, B_1 and C_1 are connected to form an enlarged triangle $A_1B_1C_1$.

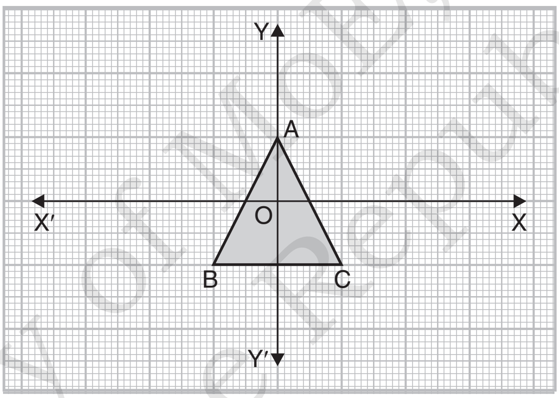
Note: In an enlargement there is a centre of enlargement and a scale factor.

Example 7. *Draw the enlargements of the following figures (figure (a) and figure (b)) with O as a centre of enlargement*

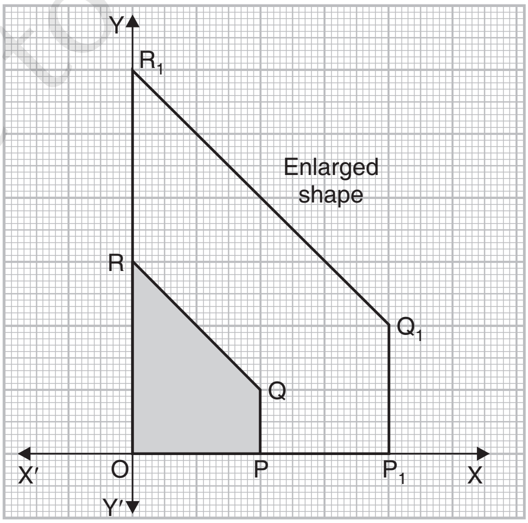
(a) *Use a scale factor of 2.*



(b) *Use a scale factor of 3.*



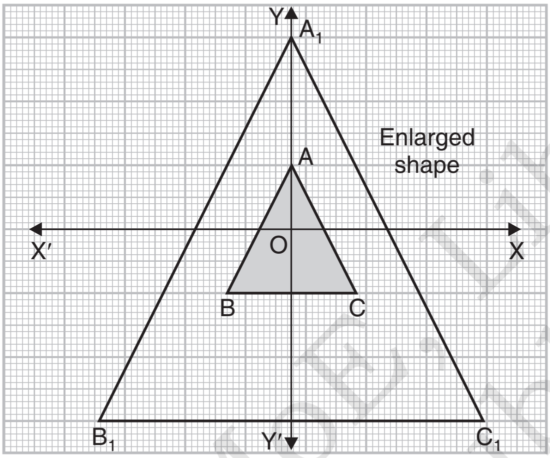
Solution. (a) The enlargement $OP_1Q_1R_1$ of the given shape $OPQR$ with O as a centre of enlargement, using a scale factor of 2 is shown in figure.



For a scale factor of 2,

$$OP_1 = 2 \times OP,\ OQ_1 = 2 \times OQ,\ OR_1 = 2 \times OR$$

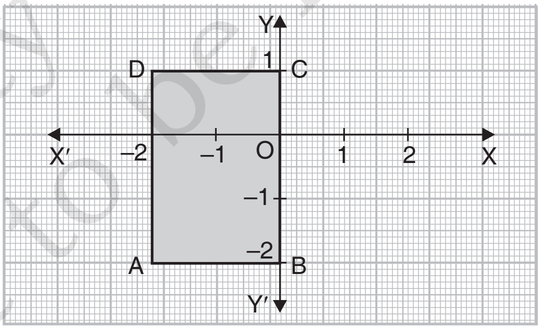
(b) The enlargement $A_1B_1C_1$ of the given shape ABC with O as a centre of enlargement, using a scale factor of 3 is shown in figure.



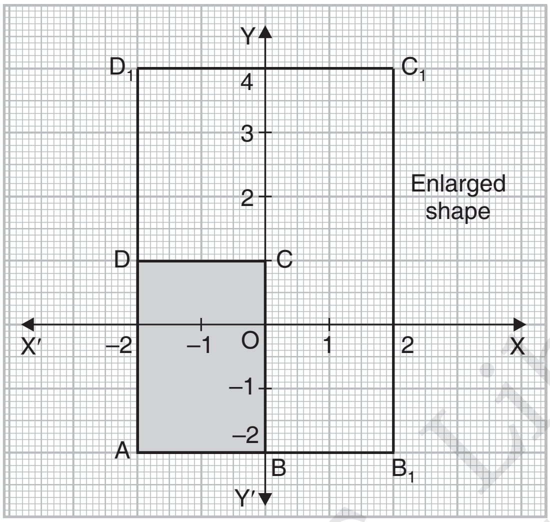
For a scale factor of 3,

$$OA_1 = 3OA,\ OB_1 = 3OB,\ OC_1 = 3OC.$$

Example 8. *Draw an enlargement of rectangle ABCD (Fig. 51) with the centre of enlargement A and a scale factor of 2.*



Solution. The enlargement $AB_1C_1D_1$ of the given shape $ABCD$ with A as the centre of enlargement and a scale factor of 2 is shown in figure.



For a scale factor of 2,

$$AB_1 = 2AB,\ AC_1 = 2AC,\ AD_1 = 2AD$$

Stating single transformation that maps one shape onto another

Look carefully at the following figure:

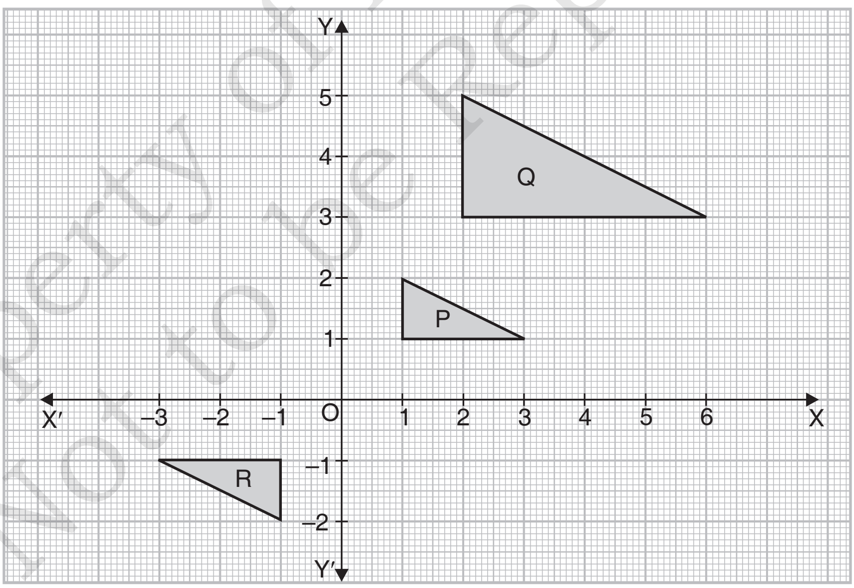


Fig. 53

There is a single transformation that:

(i) maps triangle P onto triangle Q. (see figure)

In the figure, the single movement or transformation that takes the point (1, 1) of triangle P to the point (2, 3) of triangle Q is translated by the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and using a scale factor of 2.

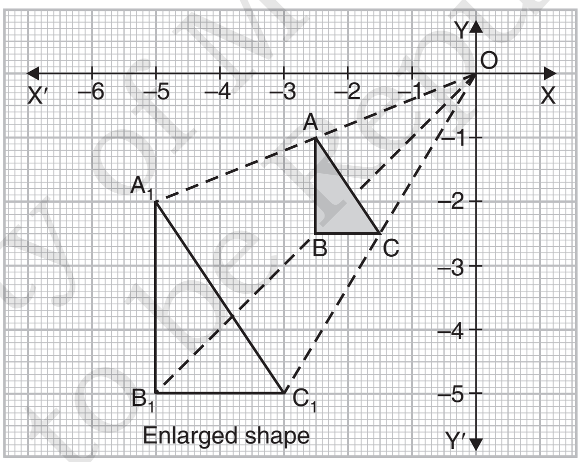
(ii) maps triangle P onto triangle R (see figure).

In the figure, the single transformation that maps triangle P onto triangle R is $(x, y) \rightarrow (-x, -y)$.

Finding the Scale Factor

In this section, we will learn about finding the scale factor by which an object is enlarged. The scale factor indicates by how much an object has been enlarged.

What is the scale factor of the following enlargement (see figure)?



The scale factor of the enlargement is 2.

Let us measure the sides of the object ABC.

Here, AB = 1.5 cm, BC = 1 cm, AC = 1.8 cm

Also the measure of the sides of the image $A_1B_1C_1$ are

A_1B_1 = 3 cm, B_1C_1 = 2 cm, A_1C_1 = 3.6 cm

Now, $\dfrac{A_1B_1}{AB} = \dfrac{B_1C_1}{BC} = \dfrac{A_1C_1}{AC} = 2.$

So, the ratio of corresponding sides is equal to the scale factor.

What do you conclude?

To find the scale factor of an enlargement, we determine the ratio of the sides of an image to the corresponding sides of the object. This ratio is the required scale factor.

Properties of Enlargement

Enlargement is also called dialation, contraction, compression, or even expansion. The enlarged shape becomes bigger or smaller.

Characteristics of Enlargements

An enlargement is a transformation that produces an image that has the *same shape* as the original, but has a *different* size. It stretches or shrinks the original figure. As discussed earlier, the description of an enlargement includes the centre of enlargement and a scale factor.

The centre of enlargement is a fixed point in the plane about which all the points are expanded or contracted. It is the only invariant point under an enlargement.

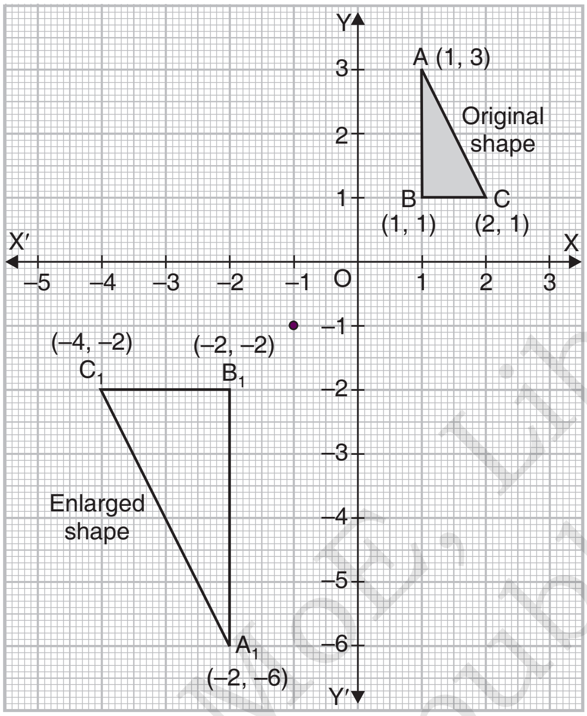
The scale factor connecting the object with its image is usually denoted by the letter K and calculated by

$$K = \frac{\text{Length of the image side}}{\text{Length of the object side}}$$

Let us investigate the characteristics of enlargements under the following conditions of the scale factor:

- ***If the scale factor (K) is negative.***

The image and the figure are on the opposite sides of the centre of enlargement. For example, the enlargement of triangle ABC with the centre of enlargement at the origin and a scale factor of –2 is shown in figure.



Observe that every coordinate of the original triangle has been multiplied by the scale factor –2.

- ***If the scale factor (K) is greater than 1 or less than –1.***

If $K > 1$, then the image is larger than the object and on the same side of the centre of enlargement (Figs.).

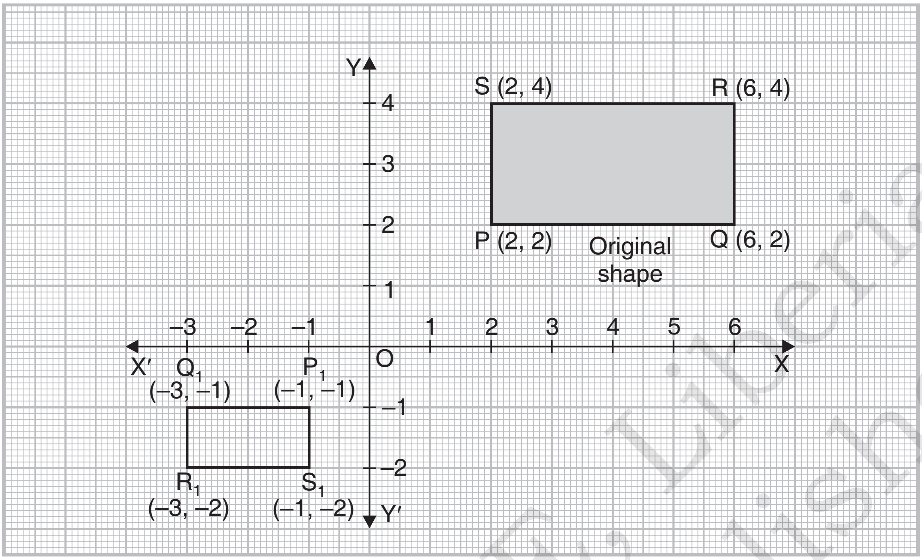
If $K < -1$, then the image is larger than the object and on the opposite side of the centre of enlargement (see figure).

- ***If the scale factor (K) is between –1 and 1 (i.e., a fraction)***

If $-1 < K < 0$, then the image is smaller than the object and on the opposite side of the centre of enlargement (see figure).

The following figure shows the enlargement of rectangle *PQRS* with the centre of enlargement at *O* and the scale factor of $-\frac{1}{2}$.

Observe that every coordinate of the original rectangle has been multiplied by the scale factor of $-\frac{1}{2}$.



If 0 < K < 1, then the image is smaller than the object.

Fig. shows the enlargement of a pentagon *OABCD* with the centre of enlargement at *O* and the scale factor of $\frac{1}{3}$.

Observe that every coordinate of the original pentagon *OABCD* has been multiplied by the scale factor $\frac{1}{3}$ to get the pentagon $OA_1B_1C_1D_1$.

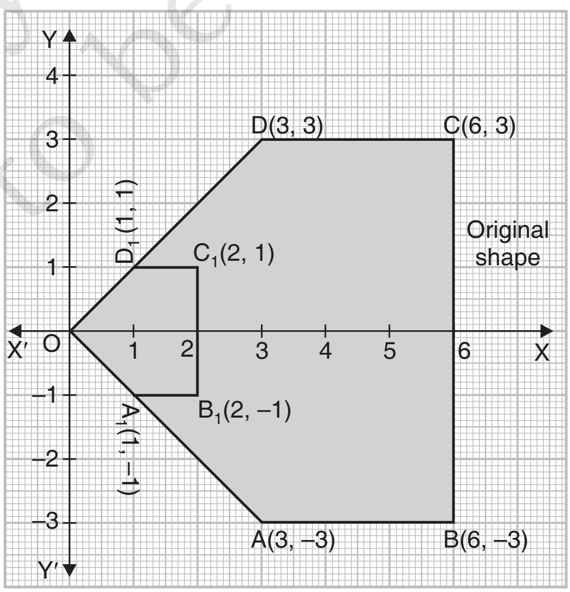


Fig. 58

10.4. SCALE DRAWING

There are many objects in the world which are too big or small for a normal sheet of paper. But we can draw these to a particular "*scale*" relative to these objects.

A *scale drawing* is a reduction or an enlargement of a plane figure that is similar to the original figure.

An example from real life: Since it is not always possible to draw on paper the actual size of real-life objects such as the real size of a car, an airplane, we need scale drawings to represent their sizes.

This is the picture of a van (Fig.).

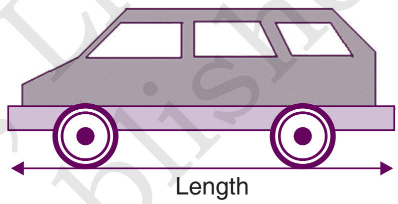


Fig.

In real-life, the length of this van may measure 6 m or 600 cm. However, the length of a copy or print paper that you could use to draw this van is a little bit less than 30 cm. Since 600 ÷ 30 = 20, you will need about 20 sheets of copy paper to draw the length of the actual size of the van.

In order to use just one sheet, you could then use 1 cm on your drawing to represent 20 cm on the real-life object.

You can write this situation as 1 : 20 or $\frac{1}{20}$ or 1 to 20.

Notice that the first number always refers to the length of the drawing on paper and the second number refers to the length of real life object.

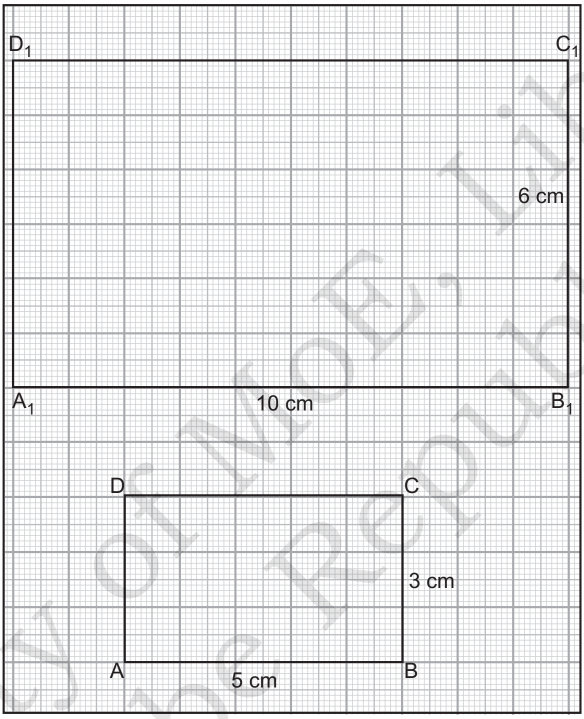
We summarize the above discussion as below.

1. A scale is the ratio of the length in a drawing (or model) to the length of the real object.

2. A **scale drawing** is a drawing that shows a real objet with accurate sizes except they have been reduced or enlarged using scale.

3. In a scale drawing, all dimensions are reduced or enlarged by the same proportion.

Scale factor: A scale factor K is simply a non-zero number that multiplies the dimensions of an object. This can make an object larger or smaller depending upon the values of K. If $K > 1$, then, multiplying the dimensions of an object by K will increase the size of the object. If $K < 1$, it will decrease the size. If $K = 1$, then the two objects are similar or congruent. Consider a rectangle $ABCD$ (Fig.) with its width as 3 cm and length as 5 cm. Now, multiplying the dimensions of this rectangle by 2, the new width is 6 cm = 2 × 3 cm and length is 10 cm = 2 × 5 cm.



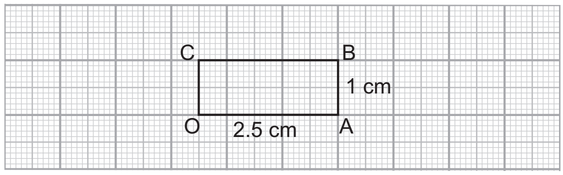
∴ The new rectangle $A_1B_1C_1D_1$ is shown in Fig., which has same shape, but not of same size. Here, the number 2 is a scale factor.

Thus, dimensions of the resulting figure = K × dimensions of the given figure, where K is the scale factor.

Note: A scale factor has no unit.

Scale factor and areas and volumes of plane figures. Given the scale factor, we can determine areas and volumes of plane figures as explained in the following examples.

Example 9. *Draw the enlargement of the following rectangle (Fig.) by using a scale of 1 : 2. Hence, find the area of the resulting rectangle.*



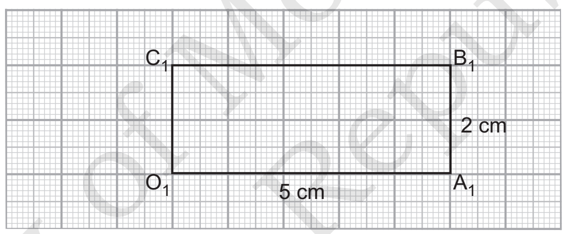
Solution. Looking at the graph, the dimensions of the rectangle are OA = 2.5 cm, AB = 1 cm.

Here the scale is 1 : 2 (from smaller rectangle to larger rectangle), therefore, scale factor = $\frac{2}{1}$ = 2.

∴ The new dimensions are

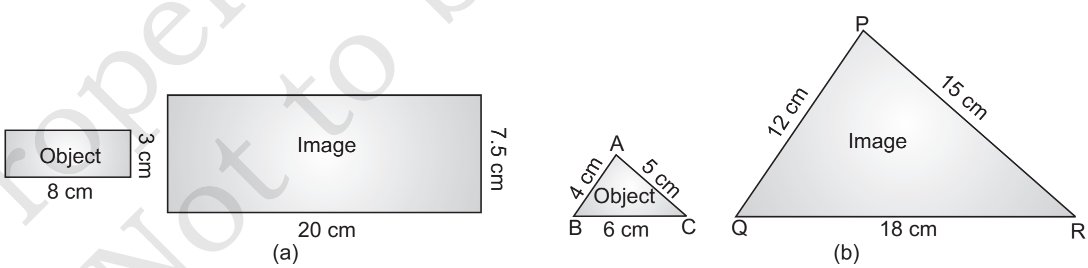
$$O_1A_1 = 2 \times 2.5 = 5 \text{ cm and}$$
$$A_1B_1 = 2 \times 1 = 2 \text{ cm}$$

The enlarged rectangle $O_1A_1B_1C_1$ is shown in Fig.



∴ Area of $O_1A_1\ B_1C_1$ = 5 × 2 = 10 cm^2

Example 10. *Find the scale factor of the following reductions (Fig. 68).*

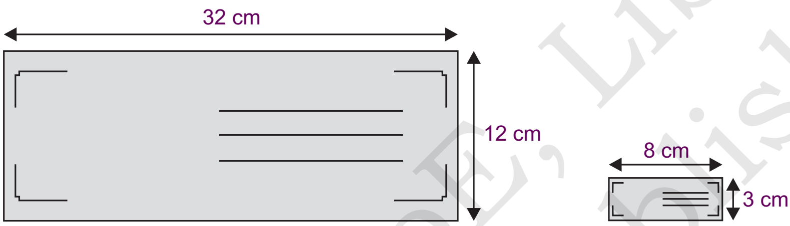


Solution. (a) We are required to find the scale factor from the larger figure (image) to the smaller figure (object).

$$\therefore \quad \text{Scale factor} = \frac{\text{Ratio of one side of object}}{\text{Ratio of the corresponding side of image}} = \frac{8}{20} = \frac{2}{5}$$

(b) Scale factor $= \frac{6}{18} = \frac{1}{3}$.

Example 11. *A design for an invitation has been scaled down using a photocopier. The original and scaled copy measurements are shown below (see figure). Find the scale factor.*



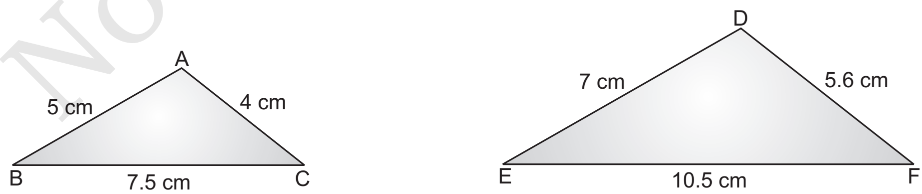
Solution. The design of the invitation has been scaled down (from larger to smaller)

$$\therefore \quad \text{Scale factor} = \frac{8}{32} = \frac{1}{4}.$$

10.5. AREAS AND VOLUMES OF SIMILAR FIGURES

"The scale factor from the first figure to the second figure is always the ratio of any one side of the second figure to the corresponding side of the first figure".

For example, the two similar triangles (Fig.) ΔABC and ΔDEF have a scale factor of $\frac{10.5}{7.5} = 1.4$ from the smaller triangle to the larger triangle.



Remember that if we write

(i) Actual dimensions of the object = K × dimension of the model, then, K will be greater than one.

(ii) Dimension of the model = K × actual dimension of the object, then, K will be less than one.

Scale Factor and Area: Consider two rectangles of dimensions 5 cm × 3 cm and 10 cm × 6 cm respectively. The scale factor is 2. Why?

Area of first rectangle = $5 \times 3 = 15 \text{ cm}^2$

Area of second rectangle = $10 \times 6 = 60 \text{ cm}^2 = 4 \times 15 \text{ cm}^2$

$= 2^2 \times$ Area of first rectangle

Thus, we can say that

Area of the resulting figure = K^2 × Area of the given figure, where K is the scale factor.

Scale Factor and Volume: Consider a rectangular prism of dimensions 4 cm × 2 cm × 3 cm. If the scale factor is 2, the new dimensions of the rectangular prism will be 8 cm × 4 cm × 6 cm. Why?

Volume of first rectangular prism = $4 \times 2 \times 3 = 24 \text{ cm}^3$

Volume of second rectangular prism

$= 8 \times 4 \times 6 = 192 \text{ cm}^3 = 8 \times 24 \text{ cm}^3$

$= 2^3 \times$ Volume of first rectangular prism

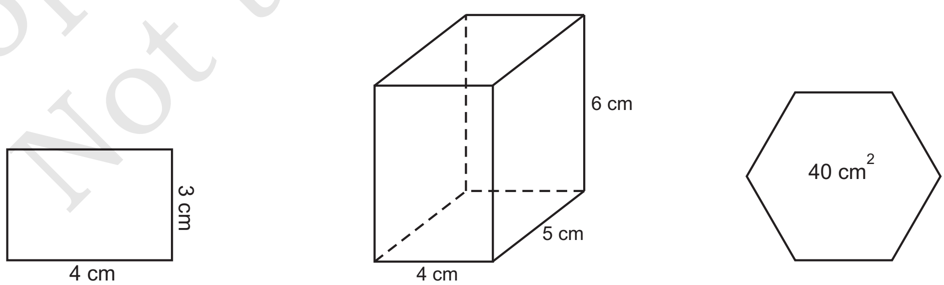
Thus, we can say that

Volume of the resulting figure

$= K^3$ × Volume of the given figure,

where K is the scale factor

Example 12. *Consider the following geometrical shapes (Fig. 76).*



(a) *What will be the area of the rectangle if it is enlarged with a scale factor 3?*

(b) *What will be the volume of the cuboid if it enlarged with a scale factor 3?*

(c) *What will be the area of the hexagon if it enlarged with a scale factor 4?*

Solution. (a) Area of the given rectangle = $4 \times 3 = 12$ cm^2

$\therefore$ Scale factor $K = 3$

$\therefore$ Area of the resulting rectangle = $3^2 \times 12 = 108$ cm^2

(b) Volume of the given cuboid

$= 4 \times 5 \times 6 = 120$ cm^3

Scale factor $K = 3$

$\therefore$ Volume of the resulting figure = $3^3 \times 120 = 3240$ cm^3

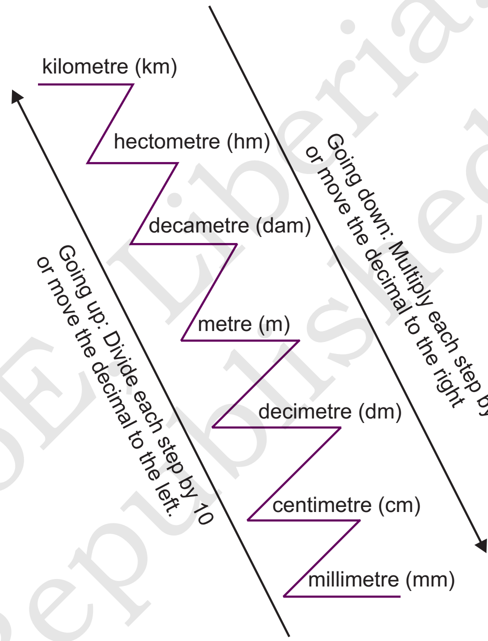
(c) Area of the given hexagon

$= 40$ cm^2

Scale factor $K = 3$

$\therefore$ Area of the resulting hexagon = $3^2 \times 40 = 360$ cm^2

We choose $K = 2 > 1$ since K is the scale factor for enlarged square.



Conversion of units.

Example 13. *A triangle with area 18 cm^2 is reduced with a scale factor $\frac{1}{2}$, find the area of reduced triangle.*

Solution. Area of reduced triangle

$= K^2 \times$ area of the given triangle

$= \frac{1}{4} \times 18 = 4.5$ cm^2

EXERCISE

1. Draw and name any one Adinkra symbol that has

(*i*) rotational symmetry of order 0

(*ii*) rotational symmetry of order 2

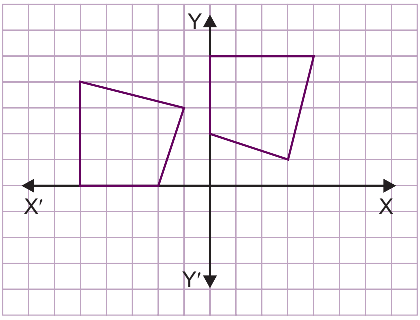
(*iii*) rotational symmetry of order 4

(*iv*) *rotational symmetry of order 12*

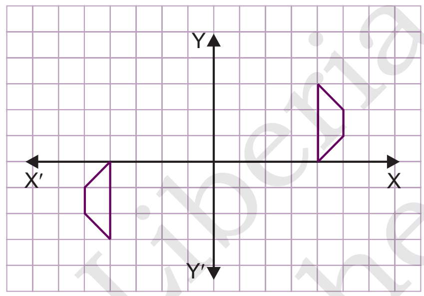
Also, find the angle of rotational symmetry and number of mirror lines in each case.

2. Below are given some geometrical shapes and their images (Fig.). Write a rule to describe each rotation.

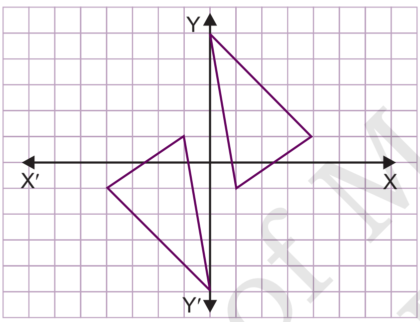
(a)



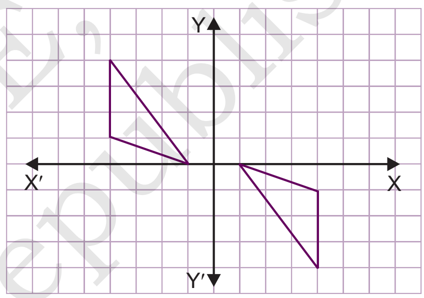
(b)



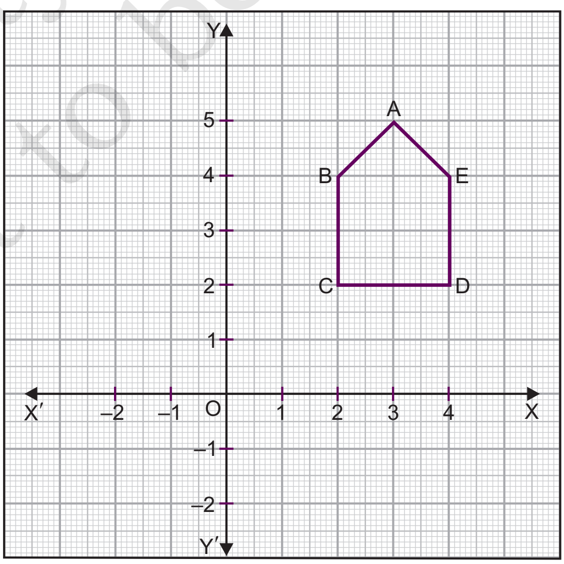
(c)



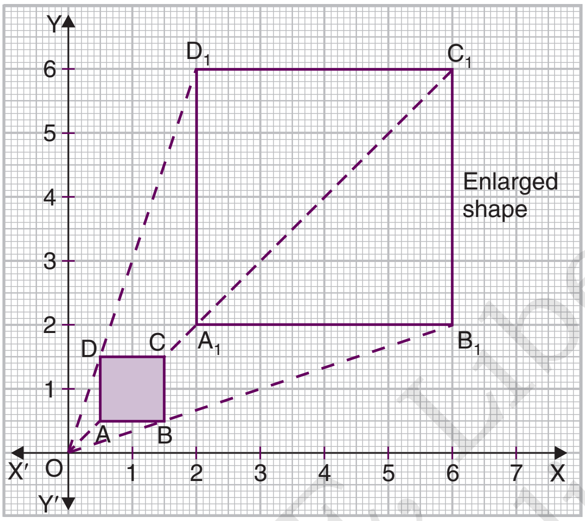
(d)



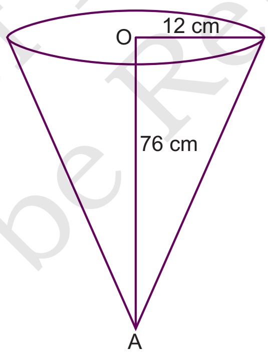
3. Rotate the pentagon ABCDE as shown in Fig. by 90° clockwise about the vertex E. Also draw the image figure.



4. Find the scale factor of the following enlargement (Fig.).



5. A cone (Figure) with diameter 12 cm and height 76 cm is to be reduced by a scalar factor of $\frac{1}{4}$. What will be the dimensions of the reduced cone? Also, find the volume of the reduced cone.



PERIOD-III

M12CH11

TOPIC 11

Trigonometry-2

11.1. GRAPH OF TRIGONOMETRIC FUNCTIONS

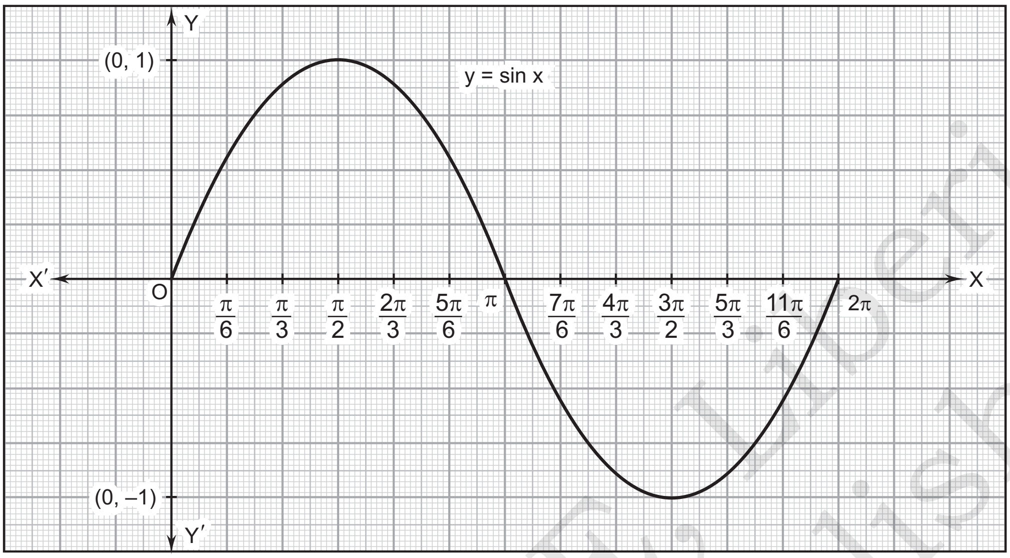
1. Graph of Sine Function

We know that $f(x) = \sin x$ is a periodic function with period 2π. Therefore, it is sufficient to know graph $f(x) = \sin x$ in the interval $[0, 2\pi]$. Using the periodicity of the function, we can draw the graph of $f(x) = \sin x$ in other intervals such as $[-2\pi, 4\pi]$ etc.

Border to draw the graph of $f(x) = \sin x$ in the interval $[0, 2\pi]$, we require the values of $\sin x$ at these points in $[0, 2\pi]$. These values are listed in the following table.

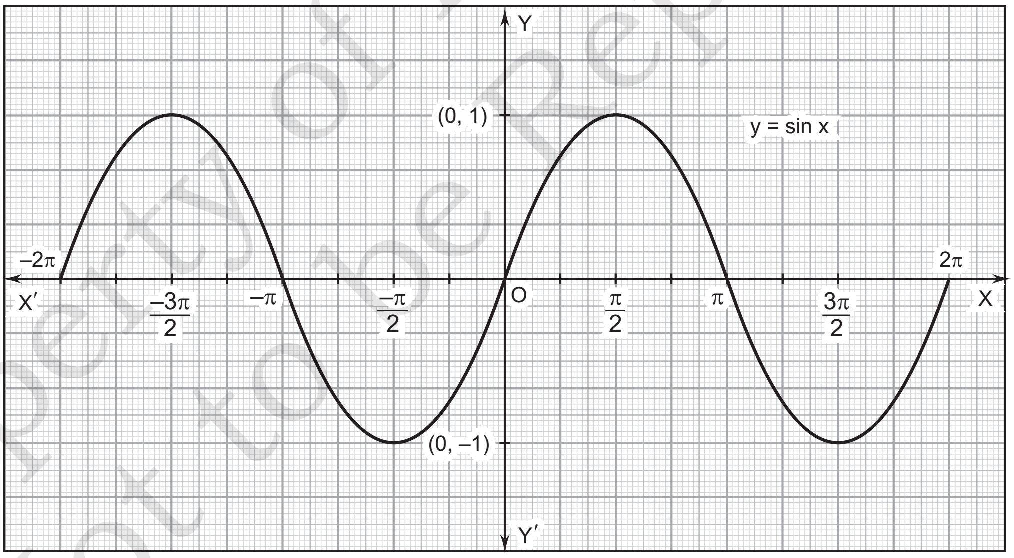
x	0	$\frac{\pi}{12}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{8}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
	$\frac{7\pi}{12}$	$\frac{5\pi}{8}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{7\pi}{8}$	$\frac{11\pi}{12}$	2π	
$y = \sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0	
$y = \sin \alpha$	0	0.5	0.71	0.86	1	0.86	0.71	0.5	0
	–0.5	–0.71	–0.86	–1	–0.86	–0.71	–0.5	0	

On suitable scale we plot the point (0, 0) ($\pi/6$, 0.5), ($\pi/4$, 0.71), ($\pi/3$, 0.86) ($\pi/2$, 1), ($2\pi/3$, 0.86), ($2\pi/4$, 0.71), ($5\pi/6$, 0.5), (π, 0), ($7\pi/6$, – 0.5), ($5\pi/4$, – 0.71), ($4\pi/3$, – 0.86) ($3\pi/3$, – 0.86), ($3\pi/2$, – 1), ($3\pi/3$, – 0.86), ($7\pi/4$, – 0.71), ($11\pi/6$, – 0.5) and (2π, 0) in the *xy*-plane and join them by a free curve to obtain the curve $y = \sin x$ *i.e.*, the graph of $f(x) = \sin x$ in the interval $[0, 2\pi]$ as shown in figure.



Graph of $f(x)$ = sin x in [0, 2x]

As $f(x)$ = sin x is a periodic function with period 2π. The graph of $f(x)$ = sin x in the interval [–2π, 0] is identical to its graph in [0, 2π] as shown in figure.



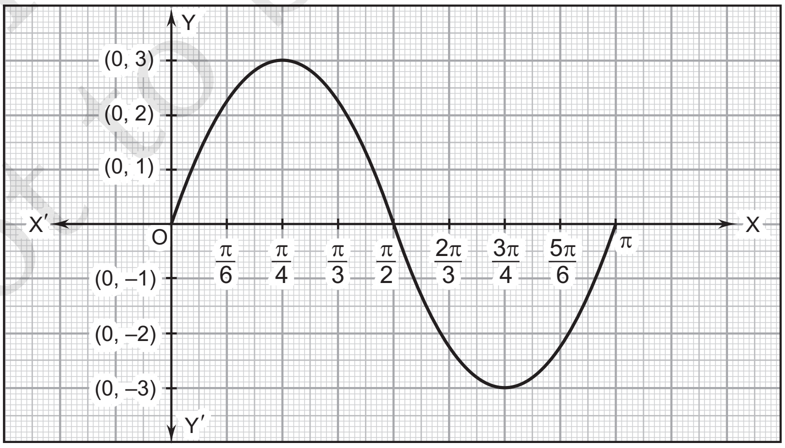
Graph of $f(x)$ = sin x in [–2π, 2π]

Example 1. *Sketch the graph of the function f(x) = 3 sin 2x.*

Solution. We know that $g(x)$ = sin x is periodic function with period 2π. Therefore $f(x)$ = 3 sin 2x is periodic funtion with period p. So, we will draw the graph of $f(x)$ = 3 sin 2x is periodic function with period π. So, we will draw the graph of $f(x) = 3 \sin 2x$ in the interval $[0, \pi]$ and to draw (or know) its graph in other intervals such as $[-\pi, 0]$, $[\pi, 2\pi]$ etc. You may use the periodicity of the function. The value of $f(x) = 3 \sin 2x$ at various points in $(0, x)$ are listed in the following table.

x	0	$\frac{\pi}{12}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{8}$	$\frac{5\pi}{12}$
	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{5\pi}{8}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{7\pi}{8}$	$\frac{11\pi}{12}$
$2x$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
$y = \sin 2x$	0	$\frac{3}{2}$ $=1.5$	$\frac{3}{\sqrt{2}}$ $= 2.1$	$\frac{3\sqrt{3}}{2}$ $= 2.59$	3	$\frac{3\sqrt{3}}{2}$ $= 2.58$	$\frac{3}{\sqrt{2}}$ $= 2.1$	$\frac{3}{2}$ $= 1.5$
	0	$-\frac{3}{2}$ $= -1.5$	$\frac{-3}{\sqrt{2}}$ $= -2.1$	$\frac{-3\sqrt{3}}{2}$ $= -2.58$	-3	$\frac{-3\sqrt{3}}{2}$ $= -2.58$	$\frac{-3}{\sqrt{2}}$ $= -2.1$	$\frac{3}{2}$ $= -1.5$

The points $(0, 0)$ $(\pi/12, 1.5)$, $(\pi/8, 2.1)$, $(\pi/6, 2.58)$, $(\pi/4, 3)$, $(\pi/3, 2.58)$, $(3\pi/8, 2.13)$ $(5\pi/12, 1.5)$, $(\pi/2, 0)$, $(7\pi/12, -1.5)$, $(5\pi/8, -2.13)$, $(2\pi/3, -2.58)$ $(3\pi/4, -3)$, $(5\pi/6, -2.58)$, $(7\pi/8, -2.13)$, $(11\pi/12, -1.5)$ and $(\pi, 0)$ are plotted on a suitable scale in the xy-plane and joined by a free hand curve to obtain the graph of the function $f(x) = 3 \sin 2x$ i.e., the curve $y = 3 \sin 2x$ as shown in figure.



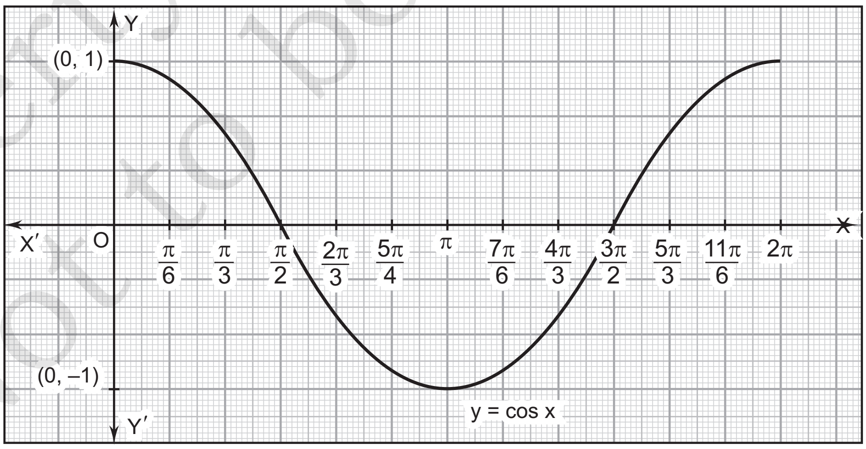
Graph of $f(x) = 3 \sin 2x$ in $[0, \pi]$

2. Graph of Cosine Function

We have learnt that $f(x) = \cos x$ is a periodic function with period 2π. In order draw the graph of $f(x) = \cos x$ it is sufficient to know its graph in $[0, 2\pi]$. The values of $f(x) = \cos x$ at various points in $[0, 2\pi]$ are given in the following table.

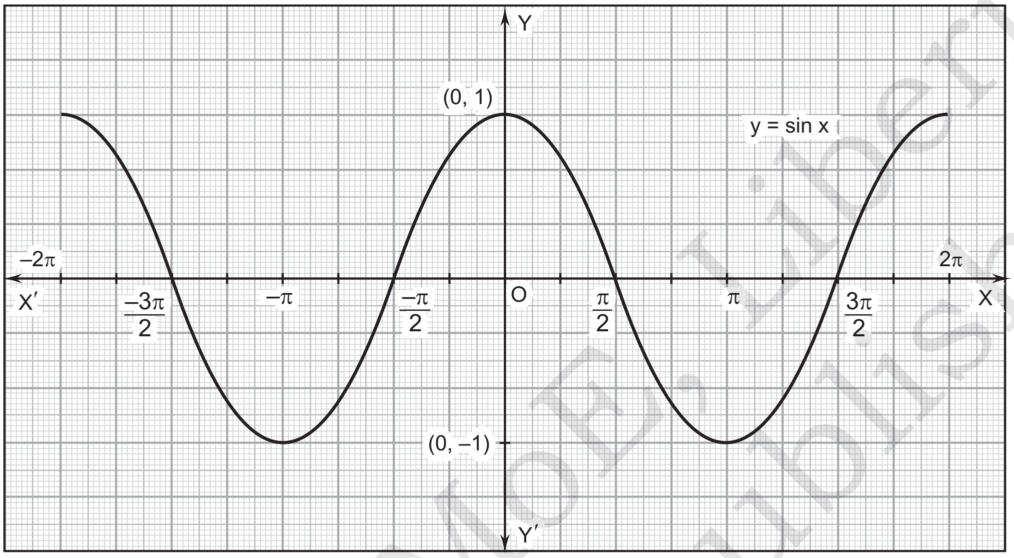
x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	
	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	
	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	1

On a suitable scale, let us plot the points $(0, 1)$, $(\pi/6, \sqrt{3}/2)$, $(\pi/4, 1/\sqrt{2})$, $(\pi/3, 1/2)$, $(\pi/2, 0)$, $(2\pi/3, -1/2)$, $(3\pi/4, -1/\sqrt{2})$, $(5\pi/6, -\sqrt{3}/2)$, $(\pi, -1)$, $(7\pi/6, -\sqrt{3}/2)$, $(5\pi/4, -1/\sqrt{2})$, $(4\pi/3, -1/2)$, $(3\pi/2, 0)$, $(5\pi/3, 1/2)$, $(7\pi/4, 1/\sqrt{2})$, $(11\pi/6, \sqrt{3}/2)$, and $(2\pi, 1)$ in the xy-plane. Now join these points by a free hand curve to obtain the graph of the function $f(x) = \cos x$ *i.e.* the curve $y = \cos x$ as shown in figure.



Graph of $f(x) = \cos x$, $0 \le x \le 2\pi$

The cosine function *i.e.* $f(x) = \cos x$ is an even function and the graph of an even function is symmetric about y-axis. So the graph of $f(x) = \cos x$ in $[-2\pi, 2\pi]$ is as shown in figure.



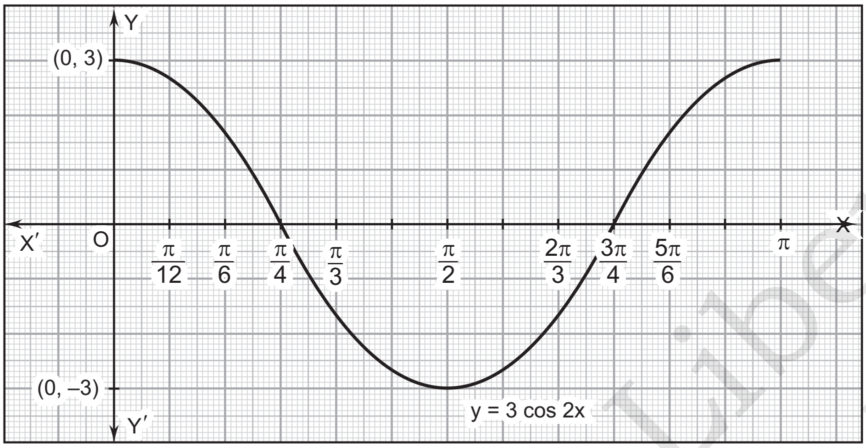
Graph of $f(x) = \cos x$, $-2\pi \le x \le 2\pi$.

Example 2. *Draw the graph of f(x) = 3 cos 2x.*

Solution. We know that cos x is a periodic function with period 2π. Therefore, $f(x) = 3\cos 2x$ is period with period π. So, it is sufficient to draw the graph of $f(x) = 3 \cos 2x$ in the interval $[0, \pi]$. The values of $3 \cos 2x$ for different values of x in $[0, x]$ are listed below.

x	0	$\frac{\pi}{12}$	$\frac{\pi}{8}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{8}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
	$\frac{7\pi}{12}$	$\frac{5\pi}{8}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{7\pi}{8}$	$\frac{11\pi}{12}$	π	
$3 \cos 2x$	3	$\frac{3\sqrt{3}}{2}$	$\frac{3}{\sqrt{2}}$	$\frac{3}{2}$	0	$-\frac{3}{2}$	$-\frac{3}{\sqrt{2}}$	$-\frac{3\sqrt{3}}{2}$	-3
	$-\frac{3\sqrt{3}}{2}$	$-\frac{3}{\sqrt{2}}$	$-\frac{3}{2}$	0	$-\frac{3}{2}$	$-\frac{3}{\sqrt{2}}$	$-\frac{3\sqrt{3}}{2}$	3	

Now, plot the points whose x-coordinates are points in the first row of the above table and the corresponding values in the second row as y-coordinates, By joining these points by a free hand curve, we obtain the graph of $f(x) = 3 \cos 2x$ as shown in figure.



Graph of $f(x) = 3 \cos 2x,\ 0 \leq x \leq \pi$

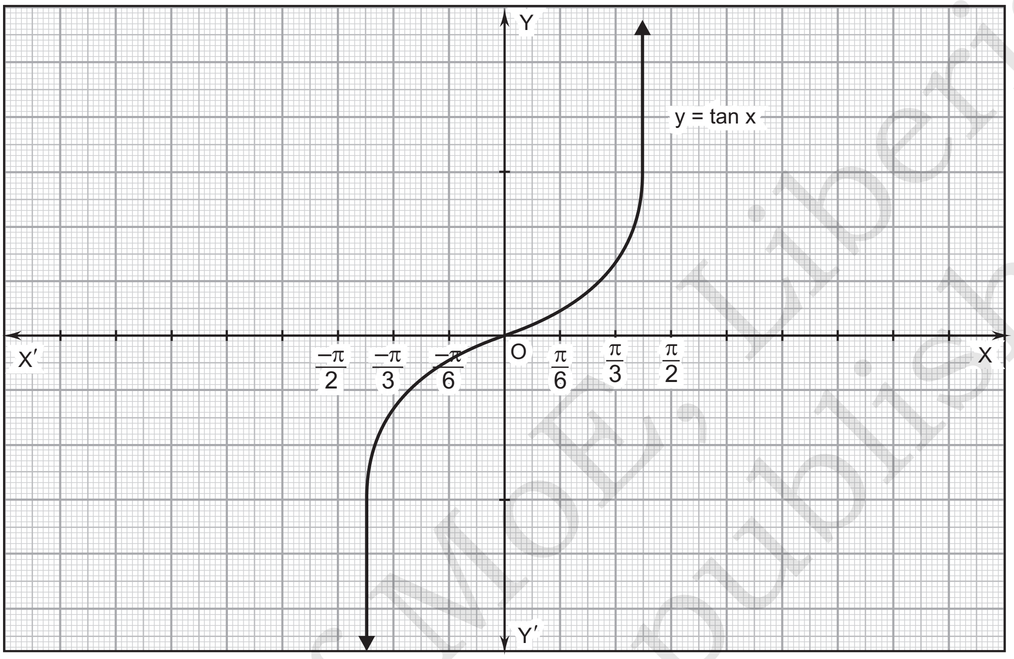
3. Graph of Tangent Function

The tangent function function *i.e.* $f(x) = \tan x$ is a periodic function with period π. So, it is sufficient to know the graph of $f(x) = \tan x$ over an interval of length π, in particular the interval $(-\pi/2, \pi/2)$. The values of $f(x) = \tan x$ at standard points in $(-\pi/2, 0)$ are negative of the corresponding values in $(0, \pi/2)$ and also listed in the following table:

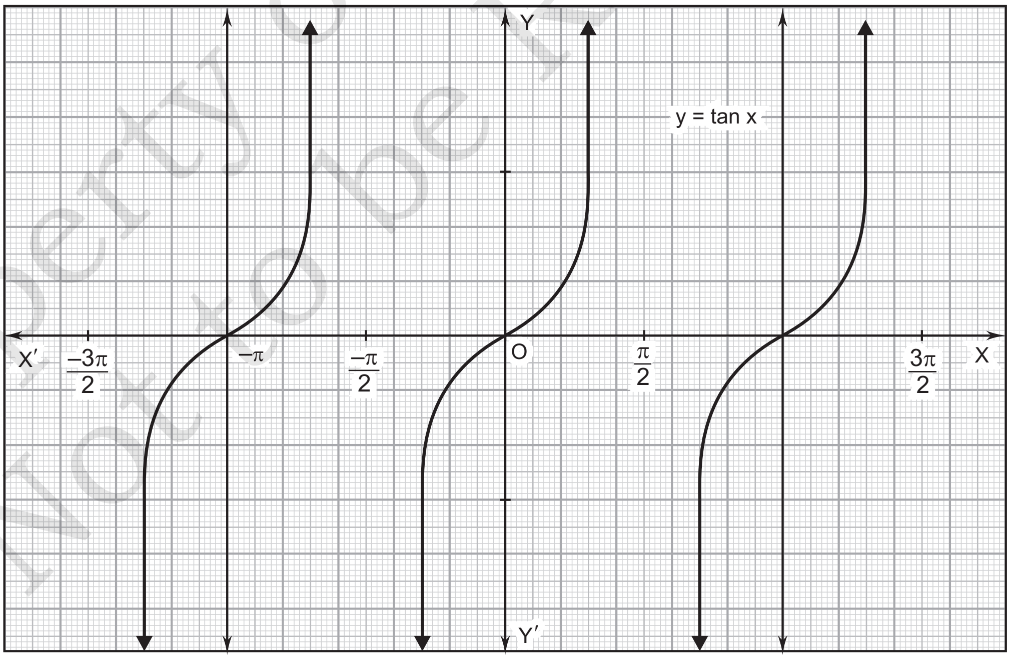
x	$-\frac{\pi}{2}$	$-\frac{5\pi}{12}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	$-\frac{\pi}{12}$	0
	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	
$f(x) = \tan x$	$-\infty$	$-\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	$-\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$	0
	$\frac{\sqrt{3}-1}{\sqrt{3}+1}$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\frac{\sqrt{3}+1}{\sqrt{3}-1}$	∞	

We also observe that $\tan x$ is an increasing function in $(0, \pi/2)$ and as $x \to \frac{\pi}{2}$ from the left the values of $f(x) = \tan x$ tend to infinity. So, the curve $y = \tan x$ gets closer and closer to the line values of $f(x) = \tan x$ tend to infinity. So, the curve $y = \tan x$ gets closer and closer to the line $x = \frac{\pi}{2}$ as $\to \frac{\pi}{2}$ from the left but it never touches the line $x = \frac{\pi}{2}$. The graph of $f(x) = \tan x$ is symmetric in opposite quadrants as the function is an add function. By plotting the points $(\pi/3, -\sqrt{3})$ $(-\pi/4, -1)$,

$(-\pi/6,\ -1/\sqrt{3})$, $(0,\ 0)$ $(\pi/6,\ 1/\sqrt{3})$, $(\pi/4,\ 1)$, $(\pi/3,\ \sqrt{3})$ and joining by a free hand curve, we obtain the sketch of the curve $y = \tan x$ as shown in figure.



Graph of $f(x) = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$



Graph of $f(x) = \tan x,\ -\frac{3\pi}{2} < x < \frac{3\pi}{2}$

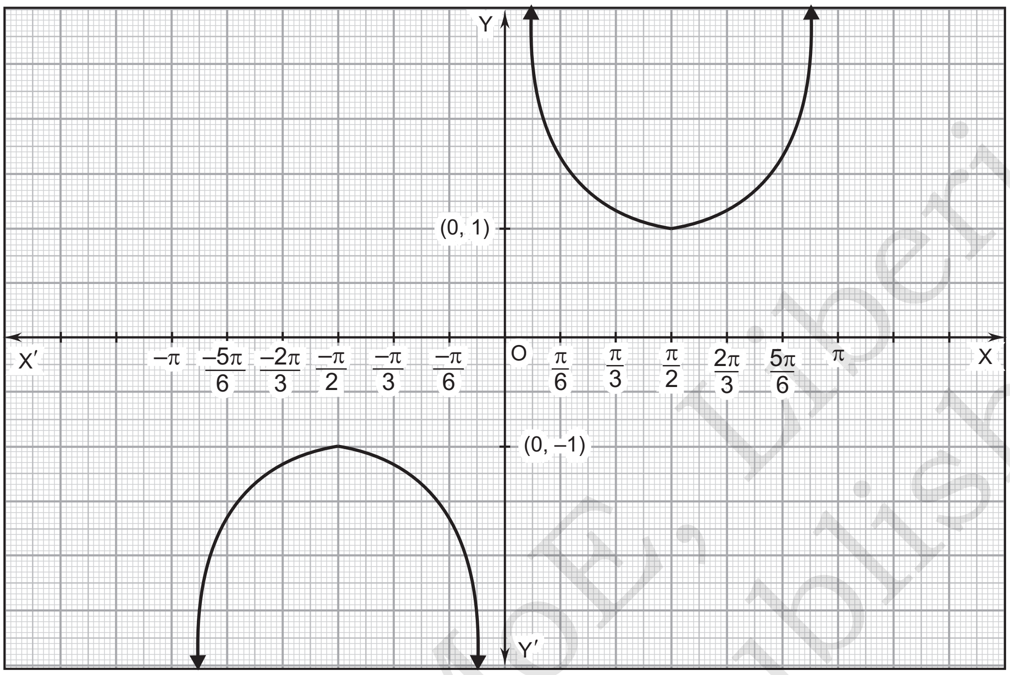
The function $f(x)$ = tan x is a periodic function with period π. So, the graph of $f(x)$ = tan x on ($\pi/2$, $3\pi/2$) and ($-3\pi/2$, $-\pi/2$) is same as its graph on ($-\pi/2$, $\pi/2$) as shown in figure.

4. Graph of Cosecant Function

The cosecant function is the reciprocal of the sine function which is periodic with period 2π. So, $f(x)$ = cosec x is periodic with period 2π. Also, $f(x)$ is defined for all $x \in$ R – {$n\pi$; $n \in$ Z}. In order to know about the graph of $f(x)$ = cosec x, it is sufficient to draw it on an interval of length 2π. Let us choose [0, 2π] as interval. The values of $f(x)$ = – cosec x at some standard points in [0, 2π] are listed in the following table. We observe that when x is close to zero or π in (0, π) the values of $f(x)$ tend to infinity. When $x \to \pi$ or $x \to 2\pi$ in (π, 2π) the values of $f(x) \to -\infty$.

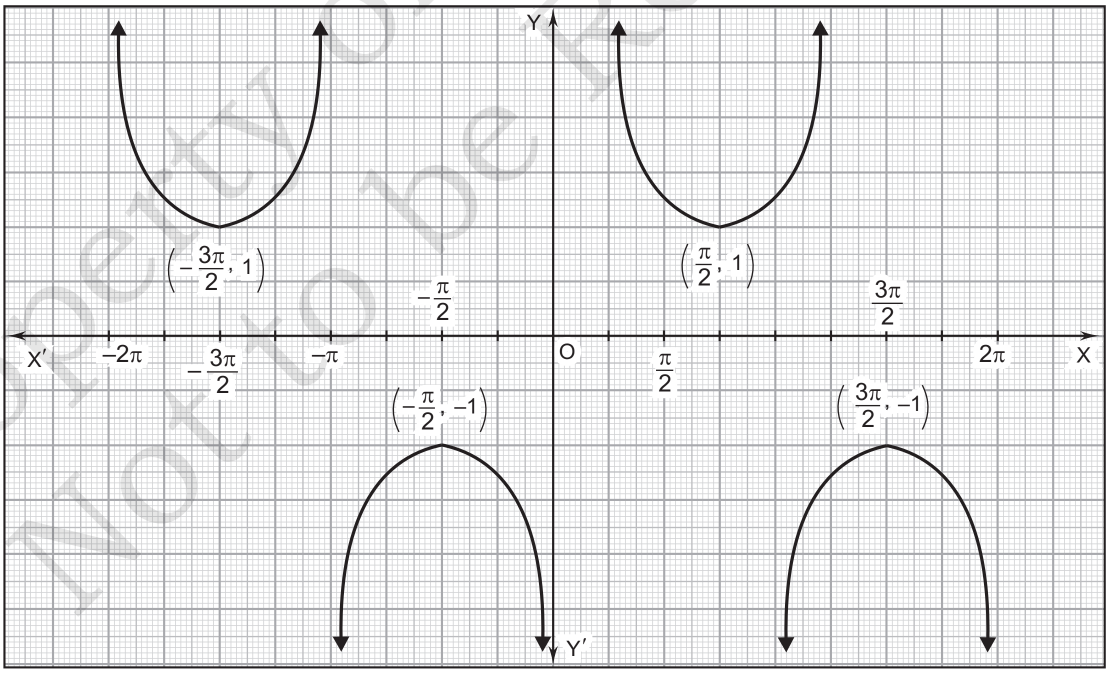
x	0^+	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$
	$\frac{5\pi}{6}$	π^-	π^+	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
$f(x)$ = cosec x	$\to\infty$	2	$\sqrt{2}$ =1.41	$\frac{2}{\sqrt{3}}$ =1.5	1 =1.15	$\frac{2}{\sqrt{3}}$ =1.41	$\sqrt{2}$
	2	$\to\infty$	$\to-\infty$	–2	$-\sqrt{2}$ =–1.41	–1.15	–1

By plotting points ($\pi/6$, 2), ($\pi/4$, $\sqrt{2}$), ($\pi/3$, $2\sqrt{3}$), ($\pi/2$, 1), ($2\pi/3$, $2\sqrt{3}$), ($3\pi/4$, $\sqrt{2}$), ($5\pi/6$, 2), ($7\pi/6$, –2), ($5\pi/4$, $-\sqrt{2}$), ($4\pi/3$, $-2\sqrt{3}$), ($3\pi/2$, –1) ($5\pi/3$, $-2\sqrt{3}$), ($7\pi/4$, $-\sqrt{2}$), ($11\pi/6$, –2) and following these observations, we obtain the graph of the function $f(x)$ = cosec x *i.e.* the curve y = cosec x as shown in figure.



Graph of $f(x) = \text{cosec } x$, $-\pi < x < \pi$, $x \neq 0$

The function $f(x) = \text{cosec } x$ is a periodic function with periodic 2π. So, the graph of $f(x) = \text{cosec } x$ in the travel $[-2\pi, 2\pi]$ is as shown in figure.

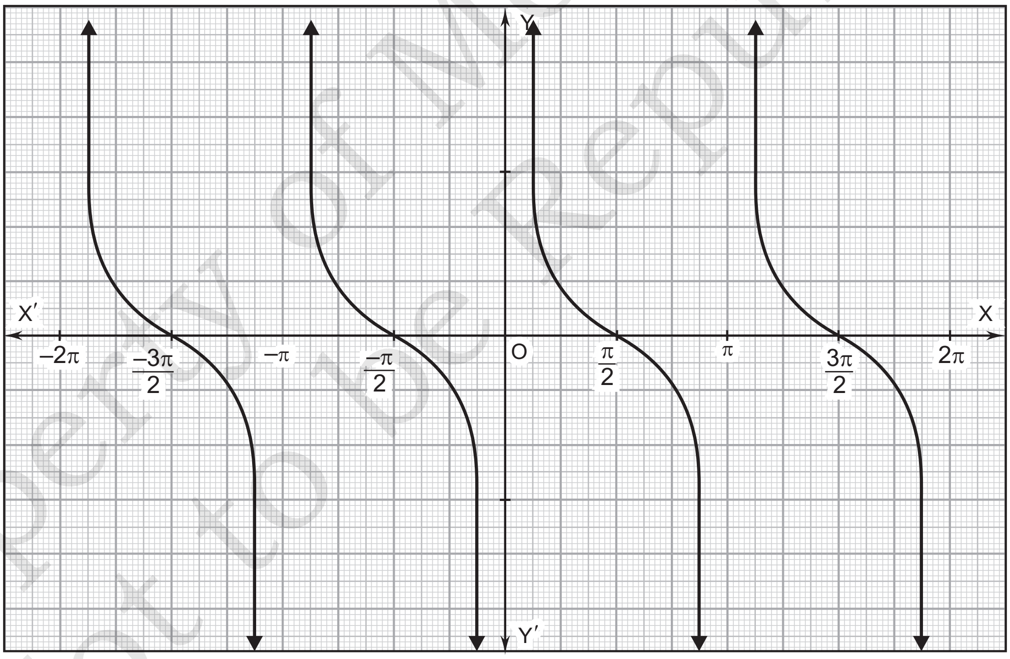


Graph of $y = \text{cosec } x$, $-2\pi < x < 2\pi$

5. Graph of Contagent Functions

We have learnt that the cotangent function i.e. $f(x)$ = cot x is a periodic function with period π. So, it is sufficient to know the graph of $f(x)$ = cot x over an interval of length π, in particular the interval (0, π). The values of $f(x)$ = cot x at some standard values of x in (0, π) are listed in the following table. We also observe the cot x is decreasing function in (0, π) and as $x \to 0^-$ the values of cot $x \to +\infty$. So, the curve y = cot x gets closer and closer to the line $x = 0$ i.e. y-axis as $x \to 0^+$. We also observe that cot $x \to -\infty$ as $x \to \pi^-$ which means that the curve y = cot x gets closer and closer to the line $x = \pi$ as $x \to \pi$ from left hand side.

By plotting the values of $f(x)$ = cot x at various points in (0, x) and keeping in mind the above observations, we obtain the graph of $f(x)$ = cot x in (0, π) as shown in figure. As $f(x)$ = cot x is periodic function with period π so the graphs of $f(x)$ = cot x in (–2π, –π), (–π, 0) and (π, 2π) are similar to the curve y = cot x in (0, π) as shown in figure.



Graph of y = cot x, $-2\pi < x < 2\pi$

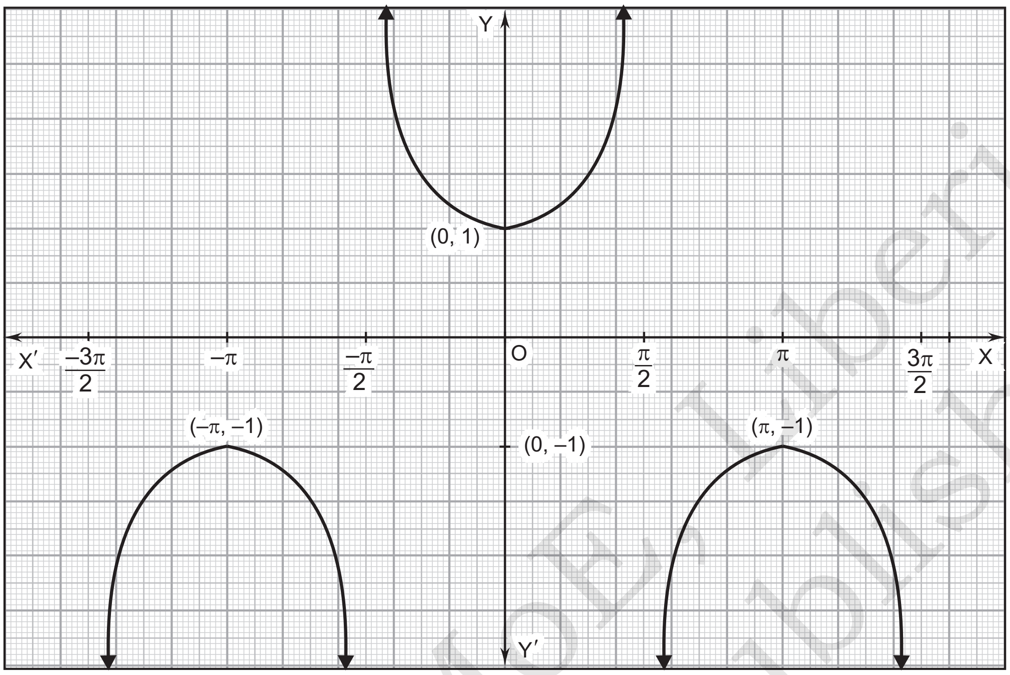
x	0^+	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π^-
	π^+	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$y = \cos x$	$\to\infty$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	$-\infty$
	$+\infty$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	$-\infty$

6. Graph of Secant Function

Similar to the other trigonometric functions the secant function is also a periodic function with period π. In order to know that graph of the secant function i.e. $f(x) = \sec x$, it is sufficient to draw it in an interval of length π, in particular the interval $(-\pi/2, \pi/2)$. We observe that the value of $f(x)$ tend to infinity as $x \to -\pi/2$ from right hand side. So, the graph of $f(x) = \sec x$ come closer and closer to $x = -\pi/2$ and $x = \pi/2$ but it never touches them. The value of $f(x) = \sec x$ at some standard points in $(-\pi/2, \pi/2)$ are listed in the following table.

x	$-\frac{\pi^+}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
	$\frac{\pi}{3}$	$\frac{\pi^-}{2}$	$\frac{\pi^+}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$f(x) = \sec x$	$\to\infty$	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$
	2	∞	$-\infty$	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1

By plotting the points given by the above table and joining them by a free hand curve. We obtains the graph of $f(x) = \sec x$ *i.e.*, the curve $y = \sec x$ as shown in figure.



Graph of $y = \sec x,\ -\frac{3\pi}{2} < x < \frac{3\pi}{2}$

EXERCISE

1. Sketch the graphic of the function $f(x) = 3 \sin\left(2x - \frac{\pi}{4}\right)$.

2. Draw the graph of $y = 3 \cos 2x$.

3. Draw the graph of $f(x) = \cos\left(2x - \frac{\pi}{4}\right)$.

SEMESTER-II

PERIOD-IV

M12CH12

TOPIC

12 Numbers and Numeration

12.1. VARIOUS TYPES OF NUMBERS

1. **Natural numbers:** *Counting numbers are called natural numbers.*
   Thus 1, 2, 3, 4, ... are all natural numbers.
2. **Whole numbers:** *All Counting numbers, together with 0; from the set of whole numbers.*
   Thus 1, 2, 3, 4, ... are all whole numbers.
3. **Integers:** *All Counting numbers zero and negatives of counting numbers, form the set of integers.*
   Thus, ... – 3, – 2, – 1, 0, 1, 2, 3 .... are all integers.
   Set of positive integers = {1, 2, 3, 4, 5, 6, ...}
   Set of negative integers = {– 1, – 2, – 3, – 4, – 5, – 6, ...}
   Set of all non-negative integers = {0, 1, 2, 3, 4, 5, ...}
4. **Even numbers:** *A counting number divisible by 2 is called an even number.*
   Thus, 0, 2, 4, 6, 8, 10, 12, ... etc. are all even numbers.
5. **Odd numbers:** *A counting number not divisible by 2 is called an odd number.*
   Thus, 1, 3, 5, 7, 9, 11, 13, ... etc. are all odd numbers.
6. **Prime numbers:** *A counting number is called a prime numbers if it has exactly two factors, namely itself and 1.*
   Ex. All prime numbers less than 100 are
   2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

7. **Composite numbers:** *All counting numbers, which are not prime, are called composite numbers.*

   A composite numbers has more than 2 factors.

8. **Perfect numbers:** *A number, the sum of whose factors (except the number itself) is equal to the number is called a perfect numbers, e.g. 6, 28, 496.*

   The factors of 6 are 1 2, 3, and 6 And, 1 + 2 + 3 = 6.

   The factors of 28 are 1 2, 4, 7, 14 and 28 And, 1 + 2 + 4 + 7 + 14 = 28.

9. **Co-primes (or Relative numbers):** *Two numbers whose H.C.F is 1 are called co-primes numbers,*

   Ex. (2, 3), (8, 9) are pairs of co-primes.

10. **Twin primes:** *Two prime numbers whose difference is 2 are called twin-primes.*

    Ex. (3, 5), (5, 7), (11, 13) are pairs of twin-primes.

11. **Rational numbers:** *Numbers which can be expressed in the form of $\frac{p}{q}$ where p and q are integers and $q \neq 0$ are called rational numbers.*

    Ex. $\frac{1}{8}, \frac{-8}{11}, 0, 6, 5\frac{2}{3}$ etc.

12. **Irrational numbers:** *Numbers which can be expressed in decimal would be in non-terminating and non-repeating form, are called irrational numbers.*

    Ex. $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \pi, e, 0.231764735.......$

Important Facts

1. All natural numbers are whole numbers.
2. All whole numbers are not natural numbers.
3. Even number + Even number = Even number

   Odd number + Odd number = Even number

   Even number + Odd number = Odd number

   Even number – Even number = Even number

   Odd number – Odd number = Even number

Even number – Odd number = Odd number
Odd number – Even number = Odd number
Even number × Even number = Even number
Odd number × Odd number = Even number
Even number × Odd number = Even number

4. The smallest prime number is 2.
5. The only even prime number is 2.
6. The first odd prime number is 3.
7. 1 is a unique number – neither prime not composite.
8. The least composite number is 4.
9. The least odd composite number is 9.
10. Test for a number to be prime.

    Let p be a given number and let n be the smallest counting numbers such that $n^2 \geq p$.

    Now, test whether p is divisible by any of the prime numbers less than or equation to n. If yes, then p is not prime otherwise, p is prime.

Example 1. *Test, which of the following are prime numbers*?

(*i*) 137 (*ii*) 173 (*iii*) 319
(*iv*) 437 (*v*) 811

Solution. (*i*) We know that $(12)^2 > 137$

Prime numbers less than 12 are 2, 3, 5, 7, 11

Clearly, none of them divides 137.

∴ 137 is a prime number.

(*ii*) We know that $(14)^2 > 173$

Prime numbers less than 14 are 2, 3, 5, 7, 11, 13

Clearly, none of them divides 173.

∴ 173 is a prime number.

(*iii*) We know that $(18)^2 > 319$

Prime numbers less than 18 are 2, 3, 5, 7, 11, 13, 17

Out of these prime numbers, 11 divides 319 completely.

∴ 319 is not a prime number.

(*iv*) We know that $(21)^2 > 437$

Prime numbers less than 21 are 2, 3, 5, 7, 11, 13, 17, 19

Clearly, 437 is divisible by 19.

$\therefore$ 437 is not a prime number.

(*v*) We know that $(30)^2 > 811$

Prime numbers less than 30 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29

Clearly, none of these numbers divides 811.

$\therefore$ 811 is a prime number.

Important Formulae

(*i*) $(a + b)^2 = a^2 + b^2 + 2ab$

(*ii*) $(a - b)^2 = a^2 + b^2 - 2ab$

(*iii*) $(a + b)^2 (a - b)^2 = 2(a^2 + b^2)$

(*iv*) $(a + b)^2 (a - b)^2 = 4ab$

(*v*) $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

(*vi*) $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$

(*vii*) $a^2 - b^2 = (a + b)(a - b)$

(*viii*) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

(*ix*) $a^3 + b^3 = (a + b)(a^2 + b^2 - 2ab)$

(*x*) $a^3 - b^3 = (a - b)(a^2 + b^2 + ab)$

(*xi*) $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

(*xii*) If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$

1. **Divisibility by 2:** *A number is divisible by 2 if its unit digit is any of 0, 2, 4, 6, 8.*

   Ex. 58694 is divisible by 2, while 86945 is not divisible by 2.

2. **Divisibility by 3:** *A number is divisible by 3 only when the sum of its digits is divisible by 3.*

   Ex.

   (*i*) In the number 695421 the sum of digits = 27, which is divisible by 3.

   $\therefore$ 695421 is divisible by 3

   (*ii*) In the number 948653, the sum of digits = 35, which is not divisible by 3.

   $\therefore$ 948653, is not divisible by 3

3. **Divisibility by 9:** *A number is divisible by 9 only when the sum of its digits is divisible by 9.*

Ex.

(*i*) In the number 246591, the sum of digits = 27, which is divisible by 9.

$\therefore$ 246591 is divisible by 9

(*ii*) In the number 734519, the sum of digits = 29, which is not divisible by 9.

$\therefore$ 734519 is not divisible by 9.

4. **Divisibility by 4:** *A number is divisible by 4 if the number formed by its last two digits is divisible by 4.*

Ex.

(*i*) 6879376 is divisible by 4, since 76 is divisible by 4.

(*ii*) 496138 is not divisible by 4, since 38 is not divisible by 4.

5. **Divisibility by 8:** *A number is divisible by 8 if the number formed by its last three digits is divisible by 8.*

Ex.

(*i*) In the number 16789352, the number formed by last 3 digits, namely 325 is divisible by 8,

$\therefore$ 16789352 is divisible by 8.

(*ii*) In the number 576484, the number formed by last 3 digits, namely 484 is not divisible by 8,

$\therefore$ 576484 is not divisible by 8.

6. **Divisibility by 10:** *A number is divisible by 10 only when its unit digit is 0.*

Ex.

(*i*) 7849320 is divisible by 10, since its unit digit is 0.

(*ii*) 678405 is not divisible by 10, since its unit digit is not 0.

7. **Divisibility by 5:** *A number is divisible by 5 only when its unit digit is 0 or 5.*

Ex.

(*i*) Each of the numbers 76895 abd 68790 is divisible by 5.

8. **Divisibility by 11:** *A number is divisible by 11 if the difference between the sum of its digits at odd places and the sum of its digits at even places is either 0 or a number divisible by 11.*

Ex.

(*i*) Consider the number 29435417

(Sum of its digits at odd places) – (Sum of its digits at even places) = (7 + 4 + 3 + 9) – (1 + 5 + 4 + 2) = (23 – 12) = 11, which is divisible by 11.

∴ 29435417 is divisible by 11.

(*ii*) Consider the number 57463822

(Sum of its digits at odd places) – (Sum of its digits at even places) = (2 + 8 + 6 + 7) – (2 + 3 + 4 + 5) = (23 – 14) = 9, which is not divisible by 11.

∴ 57463822 is not divisible by 11.

9. Divisibility by 25: *A number is divisible by 25 if the number formed by its last two digits is either 00 or divisible by 25.*

Ex.

(*i*) In the number 63875, the number formed by last 2 digits, namely 75 is divisible by 25.

∴ 63875 is divisible by 25.

(*ii*) In the number 96445, the number formed by last 2 digits, namely 45 is divisible by 25.

∴ 96445 is not divisible by 25.

10. Divisibility by 7 or 13: *Divide the number into groups of 3 digit: (starting from right) and find the difference between the sum of the numbers in odd and even places. If the difference is 0 or divisible by 7 or 13 (as the case may be), it is divisible by 7 or 13.*

Ex.

(*i*) 4537792 → 4/537/792

(792 + 4) – 537 = 259, which is divisible by 7 but not by 13.

∴ 4537792 is divisible by 7 and not by 13.

(*ii*) 579488 → 579/488

579 – 488 = 91, which is divisible by both 7 and 13.

∴ 579488 is divisible by both 7 and 13.

11. Divisibility by 16: *A number is divisible by 16, if the number formed by its last 4 digits is divisible by 16.*

Ex.

(*i*) In the number 463776, the number formed by last 4 digits, namely 3776, is divisible by 16.

∴ 463776 is divisible by 16.

(*ii*) In the number 895684, the number formed by last 4 digits, namely 5684, is not divisible by 16.

∴ 895684 is not divisible by 16.

12. Divisibility by 6: *A number is divisible by 6, if it is divisible by both 2 and 3.*

13. Divisibility by 12: *A number is divisible by 12, if it is divisible by both 3 and 4.*

14. Divisibility by 15: *A number is divisible by 15, if it is divisible by both 3 and 5.*

15. Divisibility by 18: *A number is divisible by 18, if it is divisible by both 2 and 9.*

16. Divisibility by 14: *A number is divisible by 14, if it is divisible by both 2 and 7.*

17. Divisibility by 24: *A number is divisible by 24, if it is divisible by both 3 and 8.*

18. Divisibility by 40: *A number is divisible by 40, if it is divisible by both 5 and 8.*

19. Divisibility by 80: *A number is divisible by 80, if it is divisible by both 5 and 16.*

Note: *If a number is divisible by p as well as q, where p and q are co-primes, then the given number is divisible by pq.*

If p and q are not co-primes, then the given number need not be divisible by pq, even when it is divisible by both p and q.

Ex. 36 is divisible by both 4 and 6, but it is not divisible by (4 × 6) = 24, since 4 and 6 are not co-primes.

12.2. FACTORIAL OF A NUMBER

Let *n* be a positive integer.

Then, the continued product of first n natural numbers is called factorial n, denoted by n! or $\lfloor\underline{n}$

Thus, $n! = n(n-1)(n-2) \ldots 3.2.1$

Ex. 5! = 5 × 4 × 3 × 2 × 1 = 120

Note: 0! = 1

12.3. MODULUS OF A NUMBER

$$|x| = \begin{cases} x, & \text{when } x \geq 0 \\ -x, & \text{when } x < 0 \end{cases}$$

Ex. | – 5 | = 5, |4| = 4, |– 1| = 1, etc.

12.4. GREATEST INTEGRAL VALUE

The greatest integral value of an integer x, denoted by [x], is defined as the greatest integer not exceeding x.

Ex. [1.35] = 1, $\left[\frac{11}{4}\right] = \left[2\frac{3}{4}\right] = 2$, etc.

12.5. MULTIPLICATION BY SHORT CUT METHODS

1. **Multiplication by Distributive Law:**

   (*i*) $a \times (b + c) = a\,b + a\,c$

   (*ii*) $a \times (b - c) = a\,b - a\,c$

   Ex.

   (*i*) 567958 × 99999

   = 567958 × (100000 – 1) = 567958 × 100000 – 567958 × 1

   = (59795800000 – 597958) = 56795232042

   (*ii*) 978 × 184 + 978 × 816

   = 978 × (184 + 816) = 978 × 1000 = 978000

2. **Multiplication of a Number by 5:** Put n zeros to the right of the multiplicand and divide the number so formed by 2^n.

   Ex. $975436 \times 625 = 975436 \times 5^4 = \frac{9754360000}{16} = 609647500$

12.6. DIVISION ALGORITHM OR EUCLIDEAN ALGORITHM

If we divide a given number by another number then

Dividend = (Divisor × Quotient) + Remainder

Important Facts

1. (*i*) $(x^n - a^n)$ is divisible by $(x - a)$ for all values of n.

(*ii*) $(x^n - a^n)$ is divisible by $(x + a)$ for all even values of n.

(*iii*) $(x^n + a^n)$ is divisible by $(x + a)$ for all odd values of n.

2. To find the highest power of a prime number p in $n!$

Highest power of p in $n! = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots + \left[\frac{n}{p^r}\right]$, where $p^r \le n < p^{r+1}$

Example 2. *Simplify:*

(*i*) $\dfrac{789 \times 789 \times 789 + 211 \times 211 \times 211}{789 \times 789 - 789 \times 211 + 211 \times 211}$

(*ii*) $\dfrac{658 \times 658 \times 658 - 328 \times 328 \times 328}{658 \times 658 + 658 \times 328 + 328 \times 328}$

Solution. (*i*) Given exp. $= \dfrac{(789)^3 + (211)^3}{(789)^2 - (789 \times 211) + (211)^2} = \dfrac{a^3 + b^3}{a^2 - ab + b^2}$

(where $a = 789$ and $b = 211$)

(*ii*) Given exp. $= \dfrac{(658)^3 - (328)^3}{(658)^2 + (658 \times 328) + (328)^2} = \dfrac{a^3 - b^3}{a^2 + ab + b^2}$

(where $a = 658$ and $b = 328$)

Important Facts and Formulae

I. Square Root: If $x^2 = y$ we say that the square root of y, is x and we write, $\sqrt{y} = x$.

II. Cube Root: The cube root of a given number x is the number whose cube is x. We denote the cube root of x by $\sqrt[3]{x}$

Thus, $\sqrt[3]{x} = \sqrt[3]{2 \times 2 \times 2} = 2, \sqrt[3]{343} = \sqrt[3]{7 \times 7 \times 7} = 7$ etc.

Note: 1. $\sqrt{xy} = \sqrt{x} \times \sqrt{y}$ 2. $\sqrt{\frac{x}{y}} = \sqrt{\frac{x}{y}} = \sqrt{\frac{x}{y}} \times \sqrt{\frac{y}{y}} = \sqrt{\frac{xy}{y}}$

Example 3. *Evaluate $\sqrt{6084}$ by factorization method:*

Solution. Method; Express the given number as the product of prime factors, Now, take the product of these prime factors choosing one out of every pair of the same primes. This product gives the square root of the given number.

2	6084
2	3042
3	1521
3	507
13	169
	13

Thus, resolving 6084 into prime factors, we get

$$6084 = 2^2 \times 3^2 \times 13^2$$

$$\therefore \quad \sqrt{6084} = (2 \times 3 \times 13) = 78.$$

Important Facts and Formulae

I. Logarithm: If a is a positive real number, other than 1 and $a^m = x$, then we write: $m = \log_a x$ and we say that the value of log x to the base a is m.

Example:

(*i*) $10^3 = 1000 \Rightarrow \log_{10} 1000 = 3$

(*ii*) $3^4 = 81 \Rightarrow \log_3 81 = 4$

(*iii*) $2^{-3} = \frac{1}{8} \Rightarrow \log_2 \frac{1}{8} = -3$

(*iv*) $(.1)^2 = 0.1 \Rightarrow \log_{(.1)} 01 = 2$

II. Properties of Logarithms:

1. $\log_a (xy) = \log_a x + \log_a y$
2. $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_x x = 1$
4. $\log_a 1 = 0$
5. $\log_a (x^p) = p(\log_a x)$
6. $\log_a x = \frac{1}{\log_x a}$
7. $\log_a x = \frac{\log_b x}{\log_b a} = \frac{\log x}{\log a}$
8. $a^{\log_a x} = x$
9. $x^{\log_a y} = y^{\log_a x}$
10. $\log_a x^p = \frac{p}{q} \log_a x$

Remember: When base is not mentioned, it is taken as 10.

III. **Common Logarithms:** Logarithms to the base 10 are known as common logarithms.

IV. The logarithm of a number contains two parts, namely *characteristic* and *mantissa.*

Characteristic: The integral part of the logarithm of a number is called its *characteristic.*

Case I: When the number is greater than 1.

In this case, the characteristic is one less than the number of digits to the left of the decimal point in the given number.

Case II: When the number is less than 1.

In this case, the characteristic is one more than the number of zeros between the decimal point and the first significant digit of the number and it is negative.

Instead of – 1, – 2, etc, we write $\bar{1}$ (one bar), $\bar{2}$ (two bar) etc.

Example:

Number	*Characteristic*	*Number*	*Characteristic*
348.25	2	0.6173	$\bar{1}$
46.583	1	0.03125	$\bar{2}$
9.2193	0	0.00125	$\bar{3}$

Mantissa: The decimal part of the logarithm of a number is known as it *mantissa.* For mantissa, we look through log table.

Example 4. *Evaluate:*

(*i*) $\log_3 27$ (*ii*) $\log_7\left(\frac{1}{343}\right)$

(*iii*) $\log_{100} (0.01)$ (*iv*) $\log_8 128$

Solution. (*i*) $\log_3 27 = \log_3 3^3 = 3 \log 3^3 = 3.$

(*ii*) $\log_7\left(\frac{1}{343}\right) = \log_7\left(\frac{1}{7^3}\right) = \log_7 7 - 3 = -3 \log_7 7 = -3.$

(*iii*) Let $\log_{100}(0.01) = \log_{100}\left(\frac{1}{100}\right)$

$$= \log_{100}(100)^{-1} = -1 \log_{100} 100 = -1$$

(*iv*) Let $\log_8 128 = \log_{2^3}(2^7) = \frac{7}{3}\log_2 2 = \frac{7}{3}$

Important Facts and Formulae

I. If A can do a piece of work in n days, then A's 1 day's work = $\frac{1}{n}$

II. If A's 1 day's work = $\frac{1}{n}$, then A can finish the work in n days

III. If A is thrice as good a workman as B, then
Ratio of work done by A and B = 3 : 1
Ratio of times taken by A and B to finish a work = 1 : 3

Example 5. *If Roger can do a piece of work in 8 days and Antony can complete the same work in 5 days, in how many days will both of them together complete it?*

Solution. Roger's 1 day's work = $\frac{1}{8}$; Antony's 1 day's work = $\frac{1}{5}$.

(Roger + Antony)'s 1 day's work = $\left(\frac{1}{8}+\frac{1}{5}\right) = \frac{13}{40}$

$\therefore$ Both Roger and Antony will complete the work in $\frac{40}{13} = 3\frac{1}{3}$ days.

12.7. NUMBER OF BINARY OPERATIONS

Let S be a finite set consisting of n elements. Then, S × S has n^2 elements. Since a binary operation on S is a function from S × S to S. Therefore, the total number of binary operations on S is equation to the number of functions from S × S to S. We know that the total number of functions from a finite set A to a finite set B is $(n(B))^{n(A)}$. Therefore, the total number of binary operations on S is n^{n^2}.

For example, if S = (*a, b*) then $2^{2^2} = 2^4 = 16$ binary operations can be defined on S.

Remark: If '*' is a binary operation on a set S, then we also say that 'S' is closed with respect to '*'.

Clearly, the set E of all even integers is close with respect to addition but the set O of odd integers is not closed with respect to addition as $1 \in O$, $5 \in O$ but $1 + 5 \notin O$.

12.8. TYPES OF BINARY OPERATIONS

Consider a binary operation '*' on a set S. For any two distinct elements in S, we have

$$(a, b) \neq (b, a)$$

Since '*': $S \times S \rightarrow S$. Therefore, * (a, b) and * (b, a) *i.e.,* images of (a, b) and (b, a) under '*' may or may not be same. In other words, $a * b$ and $b * a$ may or may not be equal. Thus, it is not necessary that for a binary operation * on a set S, $a * b = b * a$ must hold for all $a, b \in S$. If $a * b = b * a$ for all $a, b \in S$, then we say that binary operation * possesses commutativity as defined below.

COMMUTATIVITY: *A binary operation '*' on a set S is said to be a commutative binary operation if* $a * b = b * a$ *for all* $a, b \in S$.

The binary operations addition (+) and multiplication (×) are commutative binary operations on Z. However, the binary operation subtraction (–) is not a commutative binary operation on Z as 3 – 2 = 2 – 3.

Clearly, $\frac{ab}{2} = \frac{ba}{2}$ for all $a, b \in Q - \{0\}$

[$\because$ Multiplication is commutative on Q – {0}]

$\therefore$ $a * b = b * a$ for all $a, b \in Q - \{0\}$

So, * is commutative on Q – {0}.

Illustration 2. *Let * be a binary operation on R, the set of all real numbers, defined by* $a * b = \sqrt{a^2 + b^2}$ *for all* $a, b \in R$. *Show that * is commutative.*

Solution. We have

$\therefore$ $a * b = \sqrt{a^2 + b^2}$ and $b * a = \sqrt{b^2 + a^2}$

for all $a, b \in$ Q – {0}

But, $\sqrt{a^2 + b^2} = \sqrt{b^2 + a^2}$ for all $a, b \in$ R

$\Rightarrow$ $a * b = b * a$ for all $a, b \in$ R

So, * is commutative on R.

ASSOCIATIVITY: *A binary operation '*' on a set S is said to be an associative binary operation, if (a* b) * c = a * (b * c)* for all $a, b \in$ S.

The binary operations of addition (+) and multiplication (×) are associative binary operation on Z. However, the binary operation subtraction (–) is not as associative binary operation on Z as (2 – 3) – 5 ≠ 2 – (3 – 5).

If S is a non-empty set, then union (∪) and intersection (∩) are both commutative and associative binary operation on P(S) the power set of set (S) as

$$A \cup B = B \cup A, A \cap B = B \cup A$$

$$(A \cup B) \cup C = A \cup (B \cup C) \text{ and } (A \cap B) \cap C$$

$$= A \cap (B \cap C) \text{ for all } A, B, C \in P(S).$$

Illustration 3. *Addition of vectors is commutative as well as associative on the set V_3 of all vectors in 3-dimensional space. However, "cross-product" is neither commutative not associative on V_3.*

Illustration 4. *Addition of matrices is commutative as well as associative binary operation on $R^{m\times n}$ (set of all m × n matrices over R). Multiplication of matrices is not commutative but it is associative on $R^{n\times n}$ (set of all square m × n matrices over R). Multiplication of matrices is not commutative but it is associative on $R^{m\times n}$ (set of all square matrices of order n over R).*

Illustration 5. *Let S denote the set of all functions from a non-empty set A to itself. Clearly, composition of functions 'o' is a binary operation on S such that*

$$fog \neq gof \text{ but } (fog)oh = fo(goh) \text{ for all } f, g, h \in S.$$

Hence, composition of functions 'o' is associative but not a commutative binary operation on S.

Illustration 6. *If the operation * is defined on the set Q of all rational numbers by the rule* $a * b = \dfrac{ab}{3}$ *for all a, b* ∈ *Q. Show that * is associative on Q.*

Solution. Let $a, b, c \in$ Q *Then*

$$(a * b) * c = \frac{ab}{3} * c = \frac{\left(\frac{ab}{3}\right)c}{3} = \frac{(ab)c}{9} \qquad \ldots(i)$$

and
$$a * (b * c) = \frac{a * \left(\frac{bc}{3}\right)}{3} = \frac{a\left(\frac{bc}{3}\right)}{3} = \frac{a(bc)}{9} \qquad \ldots(ii)$$

Since multiplication is associative on Q.

$\therefore \qquad (ab)c = a(bc)$

$\Rightarrow \qquad \dfrac{(ab)c}{9} = \dfrac{a(bc)}{9}$

$\Rightarrow \qquad (a * b) * c = a * b * c \qquad$ [By using (i) and (ii)]

Thus, $(a * b) * c = a * (b * c)$ for all $a, b, c \in$ Q. Hence, * is associative on Q.

DISTRIBUTIVITY: Let S be a non-empty set and * and '⊙' be two binary operations on S. Then '*' is said to be distributive over ⊙, if for all $a, b, c \in$ S.

$a * (b \odot c) = (a * b) \odot (a * c) \qquad$ [Left distributivity of * over ⊙]

and $\quad (b \odot c) * a = (b * a) \odot (c * a) \qquad$ [Right distributivity of * over ⊙]

The binary operation multiplication (·) on Z is distributive over the binary operation addition (+) on Z because

$$a \cdot (b + c) = a.b + a.c$$

and $\quad (b + c).a = b.a + c.a$ for all $a, b, c \in$ Z.

However, addition (+) is not distributive over multiplication (·) because

$$2 + (3 \times 5) \neq (2 + 3) \times (2 + 5)$$

If S is a non-empty set, then union (∪) is distributive over intersection (∩) on P(S), because

$$A \cup (B \cap C) = (A \cup B)\ (A \cup C) \text{ for all } A, B, C \in P(S).$$

Also, intersection (∩) is distributive over union (∪) on P(S).

EXERCISE

1. Evaluate:

   (*i*) $\log_7 1 = 0$ (*ii*) $\log_{34} 34$ (*iii*) $36^{\log 64}$ (*iv*) $\log_8 128$

2. A and B together can complete a piece of work in 15 days and B alone in 20 days, In how many day can A alone complete the work?

7. Examine whether the binary operation * defined on R by $a * b = ab + 1$ is associative or not.

4. '*' is a binary operation defined on R, the set of all real numbers, by $a * b = \sqrt{a^2 + b^2}$ for all $a, b \in$ R. Show that * is associative on R.

PERIOD-IV

M12CH13

TOPIC 13

Sets and Logic

13.1. INTRODUCTION

In our day-to-day life, we come across collections of objects of a particular type, such as days in a week, months in a year, playing cards in a pack, etc. Such collections are mathematically termed as '*sets*'. Any type of objects can be collected into a set, but set theory is applied more often to objects that are relevant to mathematics. Set theory plays a foundational role in modern mathematics. *Georg Cantor*, a German mathematician, was the founder of set theory, which he defined as:

"*A set is a gathering together into a whole of definite, distinct objects of our perception and of our thought–which are called elements of the set.*"

The elements of a set can be anything: numbers, people, letters of alphabet, other sets, and so on. In this chapter, we shall discuss some basic definitions and operations involving sets.

Sets and Their Representations

In this section, we shall define a set and discuss its representations.

A set is a well-defined collection of objects.

By the phrase '*well-defined collection*', we mean that given a set and an object, it is possible to decide whether or not the object belongs to the set. Also, the decision should not vary from person to person. These objects are called *elements* of the set. Sets are usually denoted by capital letters A, B, C, ... The elements of a set are represented by small letters a, b, c, ...

For example,

(*i*) Consider the collected of first 5 natural numbers. It is a well-defined collection of objected in the sense that we can definitely decide whether a given object belongs to this collection or not. Also, the decision does not vary from to person. We can definitely say that 2 belongs to this collection, but 6 does not.

(*ii*) Consider the collection of 11 best cricket players of the word. It is not a well-defined collection of objects in the sense that we cannot definitely decide whether a given object belongs to this collection or not. Also, the decision may vary from person to person.

Remarks: 1. The term '*Set*' is synonymous with the terms '*Collection*' and '*Class*'.

2. The term '*Element*' is synonymous with the terms '*Object*' and '*Member*'.

Let us now list some of the important and frequently used sets.

$\mathbb{R}$–set of all real numbers	$\mathbb{R}^+$–Set of all positive real numbers
$\mathbb{Z}$–set of all integers	$\mathbb{Z}^+$–Set of all positive numbers
$\mathbb{Q}$–set of all rational numbers	$\mathbb{Q}^+$–Set of all positive rational numbers
$\mathbb{N}$–set of all natural numbers	$\mathbb{C}$–Set of all complex numbers

Notations

We use the following notations and terminology while working with sets:

(*i*) Let a be an element of the set A. We say that "a belongs to A". *The Greek symbol* $\in$ *(epsilon) is used to denote the phrase 'belong to'.*

(*ii*) Let b not be an element of the set A. We say that "*b does not belong to A*" and we write "$b \notin$ A".

The symbol $\notin$ is used to denote the phrase "does not belong to".

For example,

(*i*) Let A = {1, 2, 3, 4, 5}. We can say that 2 $\in$ A, but 6 $\notin$ A.

(*ii*) Let B = {*a, e, i, o, u*}. We can say that $a \in$ B, but $b \notin$ B.

Let us not discuss various methods to represent a set.

Representation of a Set

A set be repsented in the following two forms:

1. **Roster Form/Tabular Form:** In this form, we represent a set by listing all the elements of the set, the elements are being separately by commas and enclosed within braces { }.

   For example, let A be the set of all letters in the word 'MATHS'. Then set A can be written in roster form as A = {M, A, T, H, S}.

   Remarks:

   1. The order in which the elements are listed is not important. For example, (M, A, T, H, S} and {S, M, A, T, H} denote the same set.
   2. While writing the set in roster form, an element is not generally repeated, i.e., all the elements are taken as distrinct. For example, the set of all letters in the word 'BOOK' is given by {B, O, K} and not by {B, O, O, K}.

2. **Set-bullder Form.** In this form, we describe the element of the set by using a symbol *x* which is followed by a colon ":" or vertical bar "|" (read as *such that*). Then we write the characteristic property possessed by the elements of the set and enclose the whole description within braces { }.

   The characteristic property is a wingle common property which is possessed by all the elements of the set but not by any element outside the set.

   For example, if B is the set of all even integers, then set B can be written in set-builder form as

   $$B = \{x : x = 2n,\ n \in \mathbb{Z}\} = \{x \mid x = 2n,\ n \in \mathbb{Z}\}.$$

   In set-builder form, we can use any other symbol, like *y*, *z*, etc., instead of *x*.

   Let us now consider the following examples.

Example 1. *Which of the following are sets? Jusify your answer.*

(*i*) *The collection of all the months of a year beginning with the letter J.*

(*ii*) *All prime numbers less than 11.*

(*iii*) *The collection of all even integers.*

Solution. (*i*) The collection of all the months of a year beginning with the letter J s well-defined because it will always be same for every person. So, it is a set, written as {January, June, July}.

(*ii*) The collection of all prime numbers less than 11 is well-defined because it will always be same for every person. So, it is a set, written as {2, 3, 5, 7}.

(*iii*) The collection of all even integers is well-defined because it will always be same for every person. So, it is a set, written as {..., –4, –2, 0, 2, 4, ...}.

Example 2. *Which of the following are sets? Justify your answers.*

(*i*) *The collection of all rational numbers lying between –2 and 2.*

(*ii*) *The collection of all boys in your class.*

(*iii*) *The collection of question in this chapter.*

Solution.

(*i*) The collection of all rational numbers lying between –2 and 2 is well-defined because it will always be same for every person. So, it is a set.

(*ii*) The collection of all boys in my class is well-defined because it will always be same for every person. So, it is a set.

(*iii*) The collection of questions in this chapter is well-defined because it will always be same for every person. So, it is a set.

Subsets

In this section, we shall discuss an important concept of set theory, viz., subset of a set.

If every element of A is also an element of B, then we say that 'A' is a subset of 'B' or 'A is *content* in B' and we denote it by *writing* $A \subseteq B$.

The symbol '$\subseteq$' is used to denote the phrase '*is subset of* ' or '*is contained in*'.

In notational form, we write the following:

$A \subseteq B$, if $a \in A \Rightarrow a \in B$.

The symbol '$\Rightarrow$' is used to denote the word '*implies*', so we read the above statement as "A is a subset of B, If *a* belongs to A implies a belongs to B".

If there exists at least one element of A which is not an element of B, then we say that '*A is not a subste of B*' or '*A is not contained in B*' and we denote it by writing $A \not\subseteq B$.

The symbol '$\not\subseteq$' is used to denote the phrase '*is not a subset of*' or '*is not contained in*'.

Remarks:

1. Every set A is a subset of itself, i.e., $A \subseteq A$.
2. If A = B, then $A \subseteq B$ and $B \subseteq A$.
   If $A \subseteq B$ and $B \subseteq A$, then A = B.
   Hence, $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.
   We read the above statement as "A and B are equal iff $A \subseteq B$ and $B \subseteq A$".
   The symbol '$\Leftrightarrow$' is used for two-way implication, and is usually read as '*if and only if*' (briefly written as '*iff*').
3. The empty set is a subset of every set, i.e., $\phi \subseteq A$ for every set A. This is because there is no element in ϕ which does not belong to A.
4. If A is a subset of B, then we can also say that B is *superset* of A.
5. If $A \subseteq B$ and $A \neq B$, then A is called a *proper subset* of B and we denote it by $A \subset B$.
6. If A is not a proper subset of B, then we denote it by writing $A \not\subset B$.

Subset of $\mathbb{R}$

Some of the important subsets of the set $\mathbb{R}$ of the real numbers are:

(*i*) The set of all natural numbers, $\mathbb{N} = \{1, 2, 3, ...\}$.

(*ii*) The set of all integers, $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$.

(*iii*) The set of all rational numbers,

$$\mathbb{Q} = \left\{x : x = \frac{p}{q};\ p, q \in \mathbb{Z} \text{ and } q \neq 0\right\}$$

(*iv*) The set of all irrational numbers, $\mathbb{T} = \{x : x \in \mathbb{R} \text{ and } x \notin \mathbb{Q}\}$.

Clearly, we have $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R}$, $\mathbb{T} \subseteq \mathbb{R}$, $\mathbb{N} \not\subseteq \mathbb{T}$ and $Z \not\subseteq \mathbb{T}$.

Intervals as Subsets of $\mathbb{R}$

Let us now discuss the various types of intervals which are subsets of the set $\mathbb{R}$ of all real number. Then are four types of intervals:

(*i*) **Open Interval.** Let $a, b \in \mathbb{R}$ such that $a < b$. Then, the set of all real numbers between a and b excluding both the a and b is called an *open interval from a to b*. It is denoted by (a, b).

It can be shown with the help of the following diagram:

(*ii*) **Closed Interval.** Let $a, b \in \mathbb{R}$ such that $a < b$. Then, the set of all real numbers between a and b including both the a and b is called a *closed interval from a to b*. It is denoted by (a, b).

$$[a, b] = \{x : x \in \mathbb{R} \text{ and } a \leq x \leq b\}.$$

It can be shown with the help of the following diagram:

(*iii*) **Left Closed-Right Open Interval.** Let $a, b \in \mathbb{R}$ such that $a < b$. Then, the set of all real numbers between a and b including both a but excluding b, is called a *left-closed-right open interval from a to b*. It is denoted by $[a, b)$.

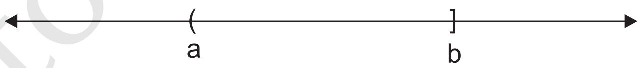
$$[a, b) = \{x : x \in \mathbb{R} \text{ and } a \leq x \leq b\}.$$

It can be shown with the help of the following diagram:

(*iv*) **Left Open Right-Closed Interval.** Let $a, b \in \mathbb{R}$ such that $a < b$. Then, the set of all real numbers between a and b excluding a but including b, is called a *left-open-right closed interval from a to b*. It is denoted by $(a, b]$.

$$(a, b] = \{x : x \in \mathbb{R} \text{ and } a \leq x \leq b\}.$$

It can be shown with the help of the following diagram:



The length of each of the interval $(a, b]$, $[a, b)$ and $(a, b]$ is $b - a$.

There are a few more types of intervals. There are as follows:

(*i*) **Open Right-Ray.** Let $a \in \mathbb{R}$. Then, the set of all real numbers greater than a is called an *open right-ray from a*. It is denoted by (a, ∞).

$$[a, \infty) = \{x : x \in \mathbb{R} \text{ and } a < x < \infty\}.$$

It can be shown with the help of the following diagram:

(*ii*) **Closed Right-Ray.** Let $a \in \mathbb{R}$. Then, the set of all real numbers greater than or equal to a is called a *closed right-ray from a.* It is denoted by (a, ∞).

$$[a, \infty) = \{x : x \leq \mathbb{R} \text{ and } a \leq x < \infty\}.$$

It can be shown with the help of the following diagram:

(*iii*) **Open Left-Ray.** Let $a \in \mathbb{R}$. Then, the set of all real numbers less than a is called an *open left-ray from a.* It is denoted by $(-\infty, a)$.

$$(-\infty, a,) = \{x : x \in \mathbb{R} \text{ and } -\infty < x < a\}.$$

It can be shown with the help of the following diagram:

(*iv*) **Closed Left-Ray.** Let $a \in \mathbb{R}$. Then, the set of all real numbers less than or equal to a is called an *closed left-ray from a.* It is denoted by $(-\infty, a)$.

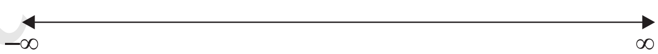
$$(-\infty, a] = \{x : x \in \mathbb{R} \text{ and } -\infty < x < a\}.$$

It can be shown with the help of the following diagram:

(*v*) **The Real Line.** Then, set of all real numbers is called *the real line.* It is denoted by $(-\infty, \infty)$.

$$(-\infty, \infty,) = \{x : x \in \mathbb{R}\}.$$

It can be shown with the help of the following diagram:



Power Set

Let A be any set. Then, we can talk about the subsets of A. Let us collect all the subsets of A to form a new set. This new set is known as *power set* of the set A. We have the following definition:

The collection of all subsets of set A is called the *power set of A.* It is denoted by P(A).

For example, consider the set A = {a, b}.

Then, ϕ{a}, {b} and {a, b} are subsets of A.

So, the power set of A is given by P(A) = {ϕ, {a}, {b}, {a, b}}.

Remarks:

1. In a power set, every elements is a set.
2. If A = ϕ, then P(A) has just one element, viz., ϕ.
3. Let A = {a, b}. Then, the set of all proper subsets of A is {ϕ, {a}, {b}}.
4. If a set A has n elements, then the number of elements in the power set P(A) is 2^n.
5. If a set A has n elements, then the number of proper subsets of A is one less than the number of elements in the power set P′(A), i.e., $2^n - 1$.

Universal Set

In any discussion in set theory, there is always a set that contains all the sets under consideration, i.e., there is always a set that is a superset of each of the sets under consideration. Such a set is called *universal set.* We have the following definition:

A set contains all sets under consideration is called *universal set.* It is denoted by U.

For example, If A = {1, 2, 3} and B = {5, 6, 8} are the sets under consideration, then the set given by U = {1, 2, 3, 4, 5, 6, 7, 8} can be taken as universal set. The choice of universal set is not unique. The set U = {1, 2, 3, 5, 6, 8} can also be taken as universal set.

Let us now understand the above concepts with the help of the following examples.

Example 3. *Let A = {a, e, i, o, u} and B = {a, b, c, d}. Is A subset of B? Is B a subset of A?*

Solution. Given that A = {a, e, i, o, u} and B = {a, b, c, d}.

Since, $e \in$ A and $e \notin$ B. So, A is not a subset of B.

Since, $b \in$ B and $b \notin$ A. So, B is not a subset of A.

Types of Sets

In this section, we shall discuss some important types of sets.

1. **Empty Set/Null Set/Void Set.** A set that does not contain any element is called the *empty set* or the *null set* or the *void set.* The empty set is denoted by the symbol ϕ (phi)or { }.
2. **Non-empty Set.** A set that contains at least one element is called a *non-empty set.* If a set A is non-empty, then we denote it by writing $A \neq \phi$.
3. **Singleton Set.** A set that contain exactly one element is called *singleton set.*

For example,

(*i*) Consider the set, $A = \{x \in \mathbb{N} : x < 0\}$. Since, there is natural number x which is negative. So, A does not contain any element. Hence, A is the empty set, i.e., $A = \phi$.

(*ii*) Consider the set, $B = \{x \in \mathbb{N} : 2 < x < 5\}$. Since, $3 \in B$, is non-empty set, *i.e.*, $B \neq \phi$.

(*iii*) Consider the set, $C = \{x \in \mathbb{N} : 2 < x < 4\}$. Then $C = \{3\}$, i.e., C is constants exactly one element. Hence, C is a singleton set.

4. **Finite Set.** A set which is empty or consists of a definite number of elements is called a *finite set.*
5. **Infinite Set.** A set which does not consists of a definite number of elements is called a *infinite set.* All infinite sets cannot be described in the roster form.

For example,

(*i*) Consider the set, $A = \{x \in \mathbb{N} : 2 < x < 3\}$. Since there is no natural number x which lies between 2 and 3. So set A does not contain any element, i.e., $A = \phi$. Hence, A is a finite set.

(*ii*) Consider the set, $B = \{x \in \mathbb{N} : 2 < x < 10\} = \{3, 4, 5, 6, 7, 8, 9\}$. Since set B contains definite number of elements (7 elements, to be precise). So, set B is a finite set.

(*iii*) Consider the set, C = *Set of all rivers in India.* Since, set C contains definite number of elements. So, set C is a finite set.

(*iv*) Consider the set, D = *Set of all natural numbers.* Since, set D does not contain definite number of elements. It can be expressed in roster form as $D = \{1, 2, 3, ...\}$.

(*v*) Consider the set, E = *Set of all real numbers.* Since, set E does not contain definite number of elements. So, set E is a infinite set. It cannot be expressed in roster form.

Remarks:

1. Empty set and singleton set are finite sets.
2. The set-builder form form of empty set is $\{x : x \neq x\}$.
3. Some people say that set $\mathbb{N}$ of all natural numbers has a definite number of elements, viz., *infinity*. This is incorrect, as *'infinity is not a number'*.

6. Cardinal Number (or order) of a Finite Set. The number of distinct elements of a finite set is called *cardinal number* or *order* of that finite set. The cardinal number of a set A is denoted by n(A). The cardinal number of the empty set is 0. The cardinal number of a singleton set is 1. The cardinal number of an infinite set is *not defined.*

For example,

(*i*) Consider the set, A = $\{x \in \mathbb{N} : 2 < x < 3\} = \{\}$. Since, A is empty set. So, n(A) = 0.

(*ii*) Consider the set, B = $\{x \in \mathbb{N} : 2 < x < 10\} = \{3, 4, 5, 6, 7, 8, 9\}$. Since, B contain 7 elements. So, n(B) = 7.

(*iii*) Consider the set, C = *set of all natural numbers* = {1, 2, 3, ...}. Since, C is an infinite set. So, n(C) is not defined.

Remark: The set of all persons in the world is a finite set, but the number of elements in the set is so large that it is very difficult to find the number. So, the cardinal number of a set is sometimes very difficult to find, but it does not mean that the cardinal number is not defined for such sets.

7. Equal Sets. Two sets A and B are said to be *equal sets* if they have exactly the same elements and we write A = B.

8. Unequal Sets. Two sets A and B are said to be *unequal sets* if they are not equal and we write A ≠ B.

For example,

(*i*) Consider the pair of sets, A = {1, 2, 3, 4} and B = {4, 3, 2, 1}. Since A and B have exactly the same elements. So, A = B.

(*ii*) Consider the pair of sets, C = {1, 2, 3, 4, 5} and D = {4, 3, 2, 1}. Since C contains 5, but D does not. So, C ≠ D.

(*iii*) Consider the pair of sets, E = {1, 2, 3, 4} and F = {1, 2, 3, 5}. Since E contains 4, but F does not. So, E ≠ F.

9. **Equivalent Sets.** Two finite sets A and B are said to be *equivalent sets* if they have the same number of elements, i.e., if $n(A) = n(B)$. We denote it by writing $A \leftrightarrow B$.

For example,

(*i*) Consider the pair of sets, A = {1, 2, 3, 4} and B = {3, 2, 4, 1}. Since, $n(A) = n(B)$. So, $A \leftrightarrow B$. We also observe that A = B.

(*ii*) Consider the pair of sets, C = {*a*, *b*, *c*} and D = {1, 2, 3}. Since, $n(C) = n(D)$. So, $C \leftrightarrow D$. We also observe that $C \neq D$.

(*iii*) Consider the pair of sets, E = {1, 2, 3, 4} and F = {*a*, *b*, *c*}. Since, $n(E) \neq n(F)$. So, E and F are not equivalent sets.

Remark: Equal sets are always equivalent but equivalent sets may or may not be equal.

Let us now understand the above types of sets with the help of following examples.

Example 4. *State which of the following sets are empty or non-empty sets:*

(*i*) *Set of all even prime numbers.*

(*ii*) *Set of all even prime numbers > 2.*

(*iii*) *Set of all circle passing through (0, 0).*

(*iv*) $\{x : 1 < x < 2, x \text{ is a natural number}\}$

(*v*) $\{x : x^2 - 2 = 0 \text{ and } x \text{ is rational number}\}$.

(*vi*) *{x : x is a point common to any two parallel lines}.*

(*vii*) $\{x : x^2 = 4, x \text{ is odd}\}$.

(*viii*) $\{x : x^2 = 4, x \text{ is odd}\}$.

Solution.

(*i*) Non-empty set (*ii*) Empty set (*iii*) Non-empty set

(*iv*) Empty set (*v*) Empty set (*vi*) Empty set

(*vii*) Empty set (*viii*) Non-empty set.

13.2. VENN DIAGRAM V/S EULER DIAGRAM

The relationships between the sets can be easily represented by means of diagrams. There are two types of representations, viz, *Venn Diagrams and Eular Diagrams.* Although both the diagrams look very similar, there are some subtle differences between them. The terms Venn diagram and Euler diagram are often confused. This section will clear the doubts about Venn diagram v/s Euler diagrams.

Similarity

The similarity between the two diagrams is that both of them consists of rectangules and closed curves usually circles). The universal set is represented by a rectangule and its subsets by circle. The interior of the circle symbollically represents the elements of the set, while the exterior represents elements that are not numbers of the set.

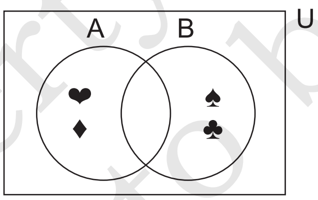
Differences

(*i*) A Venn diagram shows all possible logical relations between a finite collection of sets, whereas an Euler diagarm does not necessarily show all possible intersections of the sets.

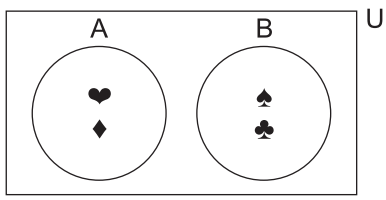
(*ii*) In a Venn diagram the circles always intersect, whereas in an Eular diagram the circles may or may not intersect.

(*iii*) In a Venn diagram two disjoint sets are represented with overlapping circles and the common region represents the empty set, whereas in an Eular diagram the disjoint sets are represented with non-overlapping circles.

Let us consider a very simple example. Let U denote the universal set of 52 playing cards. Let A and B denote the set of all red cards and the set of all black cards respectively.

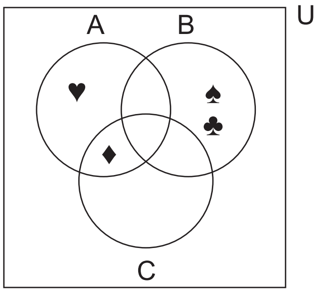


Venn diagram

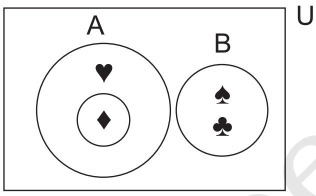


Euler diagram

Let us consider another simple example. Let U denote the universal set of 52 playing cards. Let A, B and C denote the set of all red cards, the set of all black cards and the set of all diamond cards respectively.



Venn diagram



Euler diagram

A Venn diagram shows an intersection between the two sets even though that intersection can be any empty set. Euler diagram, on the other hand, doesn't show an intersection.

However, we will restrict our presentation of subject matter to the application of Venn dagram only, as required in terms of the syllabus.

Let us now consider the following examples.

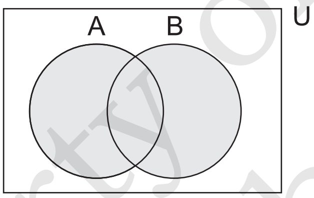
Examples 5. *Draw appropriate Venn diagram for each of the following:*

(*i*) $A \cup B$ (*ii*) $A \cap B$ (*iii*) A' (*iv*) $A - B$

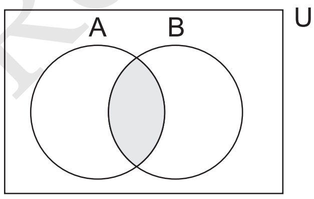
(*v*) $A \Delta B$.

Solution. Let U be the universal set. The required region is shown shaded in the graph.

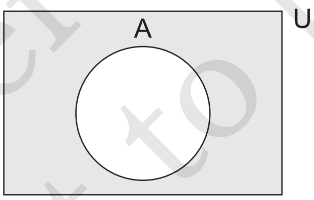
(*i*)



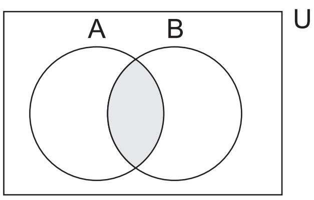
(*ii*)



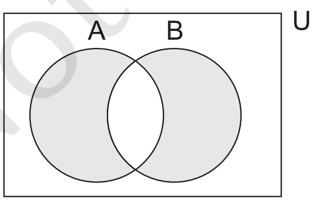
(*iii*)



(*iv*)



(*v*)



13.3. OPERATION ON SETS

We are already familier with the operations of addition, subtraction, multiplication and division of numbers. In the following sections, we shall discuss some basic operations on sets, viz., union of two sets intersection of two sets, complement of a set, difference of two sets and symmetric difference of two sets. Hence forth, we will refer all our sets as subsets of some universal set.

Union of Sets

In this section, we shall discuss the union operation on sets.

Let A and B be any two sets. Then, the *union* of A and B is the set which consists of all above elements which are earlier in A or in B or in both.

The symbol '$\cup$' is used to denote the *union* of sets.

Symbolically, we denote the union of A and B with $A \cup B$ (read as 'A *union* B') and we can write $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Some important properties of the union operation are:

1. For any set A, we have $A \cup U = U$.
2. For any set A, we have $A \cup \phi = A$.
3. (Idempotent Law) For any set A, we have $A \cup A = A$.
4. (Commutative Law) For any two sets A and B, we have $A \cup B = B \cup A$.
5. (Associative Law) For any three sets A, B and C. We have
$$A \cup (B \cup C) = (A \cup B) \cup C$$
6. For any two sets A and B such that $A \subseteq B$, we have $A \cup B = B$.

Proof:

1. $A \cup U = \{x : x \in A \text{ or } x \in U\} = \{x : x \in U\} = U$.
2. $A \cup \phi = \{x : x \in A \text{ or } x \in \phi\} = \{x : x \in A\} = A$.
3. $A \cup A = \{x : x \in A \text{ or } x \in A\} = \{x : x \in A\} = A$.
4. $A \cup B = \{x : x \in A \text{ or } x \in B\} = \{x : x \in B \text{ or } x \in A\} = B \cup A$.
5. $A \cup (B \cup G) = \{x : x \in A \text{ or } x \in B \cup C\} = \{x : x \in A \text{ or } (x \in B \text{ or } x \in C)\}$
$$= \{x : (x \in A \text{ or } x \in B) \text{ or } x \in C\}$$
$$= \{x : x \in A \cup B \text{ or } x \in C\}$$
$$= (A \cup B) \cup C$$

6. $A \cup B = \{x : x \in A \text{ or } x \in B\}$
$= \{x : x \in B\} = B$

Remarks:

1.If $x \in A \cup B$, then $x \in A$ or $x \in B$.
2.If $x \notin A \cup B$, then $x \notin A$ and $x \notin B$.
3.$A \subseteq A \cup B$ and $B \subseteq A \cup B$.
4.For there sets A, B and C, we have

$$A \cup B \cup C = \{x : x \in A \text{ or } x \in B \text{ or } x \in C\}.$$

Let us now consider the following examples.

Example 6. *Let A = {a, e, i, o, u} and B = {a, b, c}. Find A ∪ B.*

Solution. Given $A = \{a, e, i, o, u\}$ $B = \{a, b, c\}$

Then, $A \cup B = \{a, e, i, o, u\} \cup \{a, b, c\} = \{a, i, e, o, u, b, c\}$.

Intersection of Sets

In this section, we shall discuss the intersection operation on sets.

Let A and B be any two sets. Then, the *intersection* of A and B is the set which consits of all those elements which are in both A and B.

The symbol '∩' is used to denote the *intersection* of sets.

The denote the intersection of A and B with $A \cap B$ (read as '*A intersection B*') and we can write

$$A \cap B = \{x : x \in A \text{ and } x \in B).$$

Some important properties of the intersections operation are:

1. For any set A, we have $A \cap U = A$
2. For any set A, we have $A \cap \phi = \phi$.
3. (Idempotent Law) For any set A, we have $A \cap U = A$
4. (Commutative Law) For any two sets A and B, we have $A \cap B = B \cap A$.
5. (Associative Law) For any three sets A, B and C, we have $A \cap (B \cap C) = (A \cap B) \cap C$.
6. (Distributive Law) For any three sets A, B and C, we have $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
7. (Distributive Law) For any three sets A, B and C, we have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
8. For any two sets A and B such that $A \subseteq B$, we have $A \cap B = A$.

Proof:

1. $A \cap U = \{x : x \in A \text{ and } x \in U\} = \{x : x \in A\} = A.$

2. $A \cap o = \{x : x \in A \text{ and } x \in \phi\} = \phi$

3. $A \cap A = \{x : x \in A \text{ and } x \in A\} = \{x : x \in A\} = A.$

4. $A \cap B = \{x : x \in A \text{ and } x \in B\} = \{x : x \in B \text{ and } x \in A\} = B \cap A.$

5. $A \cap (B \cap U) = \{x : x \in A \text{ and } B \cap C\}$

$= \{x : x \in A \text{ and } (x \in B \text{ and } x \in C)\}$

$= \{x : (x \in A \text{ and } x \in B) \text{ and } x \in C\}$

$= \{x : x \in A \cap B \text{ and } x \in C\} = (A \cap B) \cap C.$

6. $A \cup (B \cap C) = \{x : x \in A \text{ and } B \cap C\}$

$= \{x : x \in A \text{ and } (x \in B \text{ and } x \in C)\}$

$= \{x : (x \in A \text{ and } x \in B) \text{ and } (x \in A \text{ or } x \in C)\}$

$= \{x : x \in A \cap B \text{ and } x \in C\} = (A \cap B) \cap C.$

7. $A \cap (B \cup C) = \{x : x \in A \text{ and } B \cup C\}$

$= \{x : x \in A \text{ and } (x \in B \text{ and } x \in C)\}$

$= \{x : (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)\}$

$= \{x : x \in A \cap B \text{ and } x \in A \cap C\} = (A \cap B) \cup (A \cap C).$

8. $A \cap B = \{x : x \in A \text{ and } x \in B\}$

$= \{x : x \in A\}$ $[\because \quad A \subseteq B]$

$= A.$

Remarks:

1. If $x \in A \cap B$, then $x \in A$ and $x \in B$.
2. If $x \notin A \cap B$, then $x \notin A$ or $x \notin B$.
3. $A \cap B \subseteq A$ and $A \cap B \subseteq B$.
4. For three sets A, B and C, we have $A \cap B \cap C = (x : x \in A \text{ and } x \in B \text{ and } x \in C\}$.
5. Two sets A and B are said to be *disjoint* if $A \cap B = \phi$.

Let us now consider the following examples:

Example 7. *Let A = {1, 2, 3} and B =* ϕ*. Find* $A \cap B$*.*

Solution. Given $A = \{1, 2, 3\}$ and $B = \phi = \{\,\}$.

Then, $A \cap B = \{1, 2, 3\} \cap \{\,\} = \{\,\}$.

Complement of a Set

In this section, we shall discuss the complement operation on sets.

Let U be the universal set and let A be any set. Then, the *complement of A* is the set which consists of all those elements which are in U but not in A.

Symbolically, we denote the complement of A with A′ or A^c (read as '*A complement*') and we can write

$$A' = \{x : x \in U \text{ and } x \notin A\}.$$

Some important properties of the complement operation are:

1. $\phi' = U$.
2. $U' = \phi$.
3. (Complement Laws) For any set A, we have $A \cup A' = U$ and $A \cap A' = \phi$.
4. (Double Complementation Law) For any set A, we have $(A')' = A$.
5. (De Morgan's Law) For any two sets A and B, we have

$$(A \cup B)' = A' \cap B' \text{ and } (A \cap B)' = A' \cup B'.$$

Proof:

1. $\phi' = \{x : x \in U \text{ and } x \notin \phi\} = \{x : x \in U\} = U$.
2. $U' = \{x : x \in U \text{ and } x \notin U\} = \phi$.
3. $A \cup A' = \{x : x \in A \text{ or } x \in A'\} = \{x : x \in A \text{ or } x \notin A\} = U$.
   $A \cap A' = \{x : x \in A \text{ and } x \in A'\} = \{x : x \in A \text{ or } x \notin A\} = \phi$.
4. $(A')' = \{x : x \in U \text{ and } x \notin A'\} = \{x : x \in U \text{ and } x \in A\} = A$.
5. $(A \cup B)' = \{x : x \in U \text{ and } x \notin A \cup B\}$
   $= \{x : x \in U \text{ and } (x \notin A \text{ and } x \notin B)\}$
   $= \{x : (x \in U \text{ and } x \notin A) \text{ and } (x \in U \text{ and } x \notin B)\}$
   $= \{x : x \in A' \text{ and } x \in B'\}$
   $= A' \cap B'$.

   $(A \cap B)' = \{x : x \in U \text{ and } x \notin A \cap B\}$
   $= \{x : x \in U \text{ and } (x \notin A \text{ or } x \notin B)\}$
   $= \{x : (x \in U \text{ and } x \notin A) \text{ or } (x \in U \text{ and } x \notin B)\}$
   $= \{x : x \in A' \text{ or } x \in B'\} = A' \cup B'$.

Note: The De Morgan's laws have been named after a British mathematician *Augustus De Morgan*, who has formulated these laws. These laws can be expressed as follows:

(*i*) The complement of the union of two sets is the intersection of their complements.

(*ii*) The complement of the intersection of two sets is the union of their complements.

Let us now consider the following examples.

Example 8. *Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} and A = {1, 3, 5, 7, 9}. Find A′.*

Solution. Given U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} and A = {1, 3, 5, 7, 9}. Then A′ = {2, 4, 6, 8, 10}.

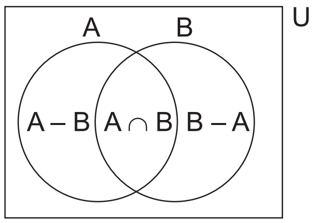
Two-Set Problems

These section. We shall discuss the problems in which two sets are under consideration. Let us now state and gave a very important theorem.

Theorem: If A and B are any two finite sets, then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Proof: Give that A and B are two finite sets. From the Venn diagram, it is clear that A – B, A ∩ B and B – A are three disjoint sets.



When the union of these three sets is equal to A ∪ B.

$$\begin{aligned} n(A \cup B) &= n(A - B) + n(A \cap B) + n(B - A) \\ &= n(A - B) + n(A \cap B) + n(B - A) + n(A \cap B) - n(A \cap B) \\ &= [n(A - B) + n(A \cap B)] + [n(B - A) + n(A \cap B)] - n(A \cap B) \\ &= n(A) + n(B) - n(A \cap B). \end{aligned}$$

this prove the theorem.

Corollary. If A and B are finite and disjoint, then

$$n(A \cup B) = n(A) + n(B).$$

In this following table, we give some verbel descriptions with their equivalent set theoretical notation (involving two sets A and B).

Verbal Description	*Set Theoretical Notation*
not A	A'
only A	$A \cap B'$
A but not B	$A \cap B'$
only B	$A' \cap B'$
B but not A	$A' \cap B$
either A or B	$A \cup B$
At least one of A and B	$A \cup B$
A and B	$A \cap B$
neither A and B	$A' \cap B'$

Let us now consider the following examples.

Example 9. *If X and Y are two sets such that X has 21 elements, Y has 32 elements, and $X \cap Y$ has 11 elements, how many elements does $X \cup Y$ have?*

Solution. We have, $n(X) = 21$, $n(Y) = 32$, $n(X \cap Y) = 11$, $n(X \cup Y) = ?$

Now, $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y) = 21 + 32 - 11 = 42$

Hence, number of elements in $X \cup Y$ is 42.

Three-Set Problems

In this section, we shall discuss the problems in which three sets are under consideration. Let us now state and prove a very important theorem.

Theorem. If A, B and C are any three finite sets, then

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C).$$

Proof. Given that A, B and C are three finite sets. Then,

$$\begin{aligned} n(A \cup B \cup C) &= n[A \cup (B \cup C)] \\ &= n(A) + n(B \cup C) - n[A \cap (B \cup C)] \\ &= n(A) + [n(B) + n(C) - n(B \cap C)] - n[A \cap (B \cup C)] \\ &= n(A) + n(B) + n(C) - n(B \cap C) - n[(A \cap B) \cup (B \cap C)] \\ &= n(A) + n(B) + n(C) - n(B \cap C) \\ &\quad - n[(A \cap B) + n \cup (A \cap C) - n(A \cap B \cap A \cap C)] \\ &= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) \\ &\quad - n(A \cap C) + n(A \cap B \cap C). \end{aligned}$$

In the following table, we give some verbal description with their equivalent set theoretical notation (involving three sets A, B and C).

Verbal Description	Set Theoretical Notation
not A	A'
only A	$A \cap B' \cap C'$
only B	$A' \cap B \cap C'$
only C	$A' \cap B' \cap C$
only A and B	$A \cap B \cap C'$
only B and C	$A' \cap B \cap C$
only A and C	$A \cap B' \cap C$
at least one of A, B and C	$A \cup B \cup C$
none of A, B and C	$A' \cap B' \cap C'$

Example 10. *In a survey of 25 students, it was found that 15 had taken Mathematics, 12 had taken Physics and 11 had taken Chemistry, 5 had taken Mathematics and Chemistry, 9 had taken Mathematics and Physics, 4 had taken Physics and Chemistry and 3 had taken all the three subjects. Find the number of students who had taken*

(*i*) *only Chemistry*
(*ii*) *only Mathematics.*
(*iii*) *only Physics*
(*iv*) *only one of the subjects.*
(*v*) *only Physics and Chemistry*
(*vi*) *only two of the subjects.*
(*vii*) *only Mathematics and Chemistry.*
(*viii*) *only two of the subjects.*
(*ix*) *at least one subject.*
(*x*) *none of the subject.*

Solution. Let U denote the set of all students,

A denote the set of students who had taken Mathematics,

B denote the set of students who had taken Physics and

C denote the set of students who had taken Chemistry.

We have, $n(U) = 25$, $n(A) = 15$, $n(B) = 12$, $n(C) = 11$, $n(A \cap C) = 5$, $n(A \cap B) = 9$, $n(B \cap C) = 4$, $n(A \cap B \cap C) = 3$.

(*i*) $n(C \cap B' \cap A')$ = ?

Now, $n(C \cap B' \cap A') = n(C) - n(C \cap B) - n(C \cap A) + n(C \cap B \cap A)$

$= 11 - 4 - 5 + 3 = 5.$

Hence, number of students who had taken only Chemistry is 5.

(*ii*) $n(A \cap C' \cap B')$ = ?

Now, $n(A \cap C' \cap B') = n(A) - n(A - C) - n(A \cap B) + n(A \cap C \cap B)$

$= 15 - 5 - 9 + 3 = 4.$

Hence, number of students who had taken only Chemistry is 4.

(*iii*) $n(B \cap C' \cap A')$ = ?

Now, $n(B \cap C' \cap A') = n(B) - n(B - C) - n(B \cap A) + n(B \cap C \cap A)$

$= 12 - 4 - 9 + 3 = 4.$

Hence, number of students who had taken only Chemistry is 2.

Now $n(C \cap B' \cap A') + n(A \cap C' \cap B') + n(B \cap C' \cap A')$ = ?

Now, $n(C \cap B' \cap A') + n(A \cap C' \cap B') + n(B \cap C' \cap A')$

$= n(A) + n(B) + n(C) - 2[n(A \cap B) + n(A \cap C) + n(B \cap C)$

$+ 3[n(A \cap B \cap C)]$

$= 15 + 12 + 11 - 2(9 + 5 + 4) + 3(3) = 11.$

Hence, number of students who had taken only one subject is 11.

Example 11. *Write the following sets in the roster form:*

(*i*) $A = \{x : x \in \mathbb{R}, 2x + 11 = 15\}$ (*ii*) $B = \{x : x^2 = x \in \mathbb{R}\}$

(*iii*) $C = \{x : x$ *is a positive factor of a prime number* $p\}$

Solution. (*i*) {2} (*ii*) {0, 1,} (*iii*) $\{1, p\}$.

Example 12. *Let* $T = \left\{x : \frac{x+5}{x-7} - 5 = \frac{4x-40}{13-x}\right\}$. *Is T an empty set? Justify your answer.*

Solution. Given $T = \left\{x : \frac{x+5}{x-7} - 5 = \frac{4x-40}{13-x}\right\}$.

Now, $\frac{x+5}{x-7} - 5 = \frac{4x-40}{13-x}$

$\Rightarrow \frac{x+5}{x-7} = \frac{4x-40}{13-x}$

$\Rightarrow (40 - 4x)(13 - x) = (4x - 40)(x - 7)$

$\Rightarrow \quad (40 - 4x)(13 - x) - (4x - 40)(x - 7) = 0$

$\Rightarrow \quad 6(40 - 4x) = 0$

$\Rightarrow \quad x = 10$

Thus, $\quad T = \left\{x : \frac{x+5}{x-7} - 5 = \frac{4x-40}{13-x}\right\} = \{10\}$

Hence, T is a non-empty set.

EXERCISE

1. Write the following in roster form:

(*i*) $\{a_n : n \in \mathbb{N}, a_{n+1} = 3a_n \text{ and } a_1 = 1\}$.

(*ii*) $\{a_n : n \in \mathbb{N}, a_{n+2} = a_{n+1} + a_n \text{ and } a_1 = a_2 = 1\}$.

2. Decide, among the following sets, which sets are subsets of one and another:

A = $\{x : x \in \mathbb{R}$ and x satisfies $x^2 - 8x + 12 = 0\}$, B = {2, 4, 6}, C = {2, 4, 6, 8, ...}, D = {6, 6}.

3. If A = {3, 5, 7, 9, 11}, B = {7, 9, 11, 13}, C = {11, 13, 15} and D = {15, 17}; find

(*i*) $A \cap B$ (*ii*) $A \cap C \cap D$. (*iii*) $A \cap (B \cup D)$

(*iv*) $(A \cap B) \cap (B \cup C)$ (*v*) $(A \cup D) \cap (B \cup C)$.

4. If Y = {1, 2, 3, ..., 10}, and a represents and element of Y, write the following sets, containing all the elements satisfying the given conditions.

(*i*) a is less than 6 and $a \in Y$ (*ii*) $a + 1 = 6, a \in Y$

(*iii*) $a \in Y$ but $a^2 \notin Y$.

5. Write the following sets in roster form:

(*i*) $\{x : x$ is an integer and $-3 < x \leq 5\}$.

(*ii*) $\{x : x$ is a two-digit natural number having sum of digits as 8$\}$.

(*iii*) $\{x : x$ is a prime number which is divisor of 60$\}$.

(*iv*) $\left\{x : x = \frac{2n+1}{n+1}, n \in \mathbb{N} \text{ and } n < 5\right\}$.

6. Write the following sets in roster form:

(*i*) The set of all even integer lying betwee – 5 and 5.

(*ii*) The set of all letters in the word 'TRIGONOMETRY'.

(*iii*) The set of all digits in our decimal system.

(*iv*) The fractions whose numerator is 3 and denominator is an even natural number less than 10.

PERIOD-IV

M12CH14

TOPIC

14 Relations and Functions, Mappings, Ratio, Proportion and Variation

14.1. RELATIONS

In this section, we shall discuss the concept of relation in detail.

Relation from a set A to set B: Let A and B be two non-empty sets. Then, a set R is said to be a relation from set A to set B if R is a subset of A × B *i.e.*, if $R \subseteq A \times B$.

Consider the two sets A = {2, 3} and B = {6. 9, 12}. The Cartesian product of A and B has 6 ordered sets which can be listed as

$$A \times B = \{(2, 6), (2, 9), (2, 12), (3, 6), (3, 9). (3, 12)\}$$

We can now obtain a subset of A × B by introducing a relation R between the first element x and the second element y of each ordered pair (x, y) as

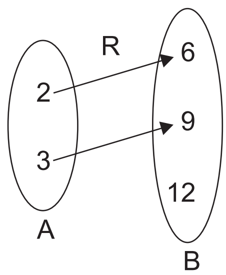
$$R = \{(x, y) : x \in A, y \in B, y \in 3x\},$$

This representation is called set-builder form of the relation R.

Also, we can write R = {(2, 6), (3, 9)}

Thus representation is called *roster form* of the relation R.

The relation R can be shown with the help of the following visual representation:



This representation is called *arrow diagram* of the relation R.

Remarks:

1. Let R be a relation from a non-empty set A to non-empty set B.
   If $(a, b) \in R$, then we say that 'a R b' *i.e.* 'a is related to b by relation R'
   If $(a, b) \notin R$, then we say that '$a \not R\ b$' *i.e.* 'a is not related to b by relation R'

2. Let number of elements in set A = p and let number of elements in set B = q.

   Then, number of elements in set A × B = pq.

   Since, each relation from A to B is a subset of A × B.

   $\therefore$ Number of relations from A to B = Number of subsets of A × B = 2^{pq}.

   Number of relations from A to B = 2^{pq}

3. Let A be any non-empty set. Then, a set R is said to be a relation on set A if R is a subset of A × A, *i.e.,* if R $\subseteq$ A × A

Example 1. *To R = {(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)} a relation from set A to set B where A = {1, 2, 3, 4} and B = {1, 5, 9, 11, 15, 16}? Justify your answer.*

Solution. Given A = {1, 2, 3, 4}, B = {1, 5, 9, 11, 15, 16}

Since R = {(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)} $\subseteq$ A × B

So, R is relation from A to B.

Function or Mapping

Let A and B be two non-empty sets. A relation f from A to B *i.e.,* a subset of A × B is called a function or mapping or a map from A to B, if

(*i*) For each $a \in$ A there exists $b \in$ B such that $(a, b) \in f$.

(*ii*) $(a, b) \in f$ and $(a, c) \in f \;\Rightarrow\; b = c$

Thus, a non-void subset of A × B is a function from A to B if each element of A appears in same ordered pair in f and no two ordered pairs in f have the same first element.

If $(a, b) \in f$ then b is called the image of a under f.

14.2. KINDS OF FUNCTIONS

If A $\rightarrow$ B is a function, then f associates all elements of set A to elements in set B such that an element of set A is associated to a unique element of set B. Following these two conditions we may associate different elements of set A to different elements of set B or more than one element on set A may be associated to the same element of set B. Similarly, there may be some elements in B which do not have their pre-images in A or all elements in B may have their pre-images in A. Corresponding to each of these possibilities we define a type of a function as given below:

One-One Function (Injection)

Definition: A function $f : A \to B$ is said to be a one-one function or an injection that it different elements of A have different images in B.

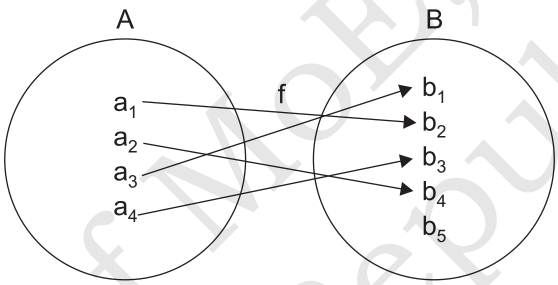
Thus, $f : A \to B$ is one-one

$\Rightarrow \quad a \neq b \Rightarrow f(a) = f(b)$ for all $a, b \in A$

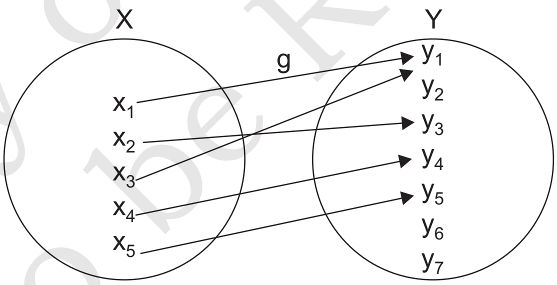
$\Rightarrow \quad f(a) = b \Rightarrow a = b$ for all $a, b \in A$

Illustration 1. *A function which associates to each country in the world, its capital is one-one because different countries have their different capitals.*

Illustration 2. *Let $f : A \to B$ and $g : X \to Y$ be two functions represented by the following diagrams:*



(a)



(b)

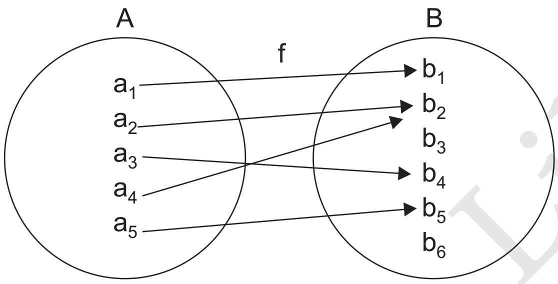
Let $f : A \to B$ is a one-one function. But $g : X \to Y$ is not one-one because two distinct elements x_1 and x_3 have the same image under function g.

Many-One Function

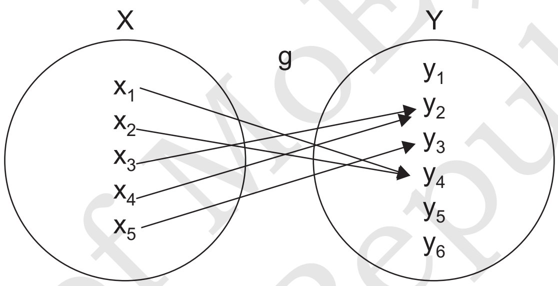
Definition: *A function $f : A \to B$ is said to be a many-one function if two or more elements of set A have the same image on B.*

Thus, $A \to B$ *is* a many-one function if there exist $x, y \in$ A such that $x = y$ but $f(x) - f(y)$.

In other words $f : A \to B$ is many-one function if it is not a one-one function.



(a)



(b)

Illustration 2. *Let* $A - \{-1, 1, -2, 2\}$ *and* $B = \{1, 4, 9, 16\}$. *Consider* $f : A \to B$ *given by* $f(x) = x^2$.

Then $f(-1)$. Thus 1 and – 1 have the same image. Similarly, 2 and – 2 also have the same image. So, f is a many-one function.

Illustration 3. *Consider a function* $f : Z \to Z$ *given by* $f(x) = |x|$ *for all* $x \in Z$. *Then f is a many-one function because for every* $a \in Z$, $a \neq 0$, *we have*

$$a \neq -a \text{ but } |a| = |-a| \Rightarrow f(a) = f(-a)$$

Illustration 4. *Show that the function* $f : Z \to Z$ *given by* $f(x) = (x)^2 + x$ *for all* $x \in Z$ *is a many one function.*

Solution. Let $x, y \in$ Z. Then,

$$f(x) = f(y)$$

$$\Rightarrow \quad x^2 + x = y^2 + y$$

$$\Rightarrow \quad (x^2 - y^2) + (x - y) = 0 \quad \Rightarrow \quad (x - y)(x - y + 1) = 0$$

$$\Rightarrow \quad x = y \text{ or } y = -x - 1$$

Since $f(x) = f(y)$ does not provide the unique solution $x = y$ but it also provides $y = x - 1$. This means that $x = y$ but $f(x) = f(y)$. When $y = -x - 1$. For example, if we put $x = 1$ in $y = -x - 1$ we obtain $y = -2$. This shows that 1 and – 2 have the same linage under *f*. Hence, *f* is a many-one solution.

14.3. ONTO FUNCTION (SURJECTION)

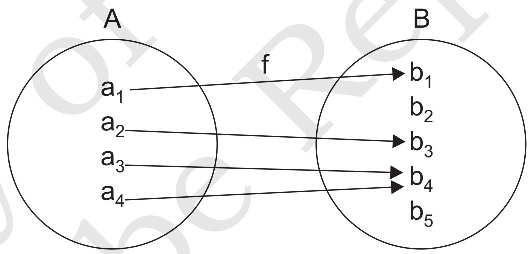
Definition: *A function* $f : A \rightarrow B$ is said to be an onto fuction or a surjection if every element of B is the *f*-image of some element of Δ *i.e.*,of *f*(A) or range of *f* is the co-domain of *f*.

Thus $f : A \rightarrow B$ is a surjection if for each $b \in B$ there exists $a \in A$ such that $f(a) = b$.

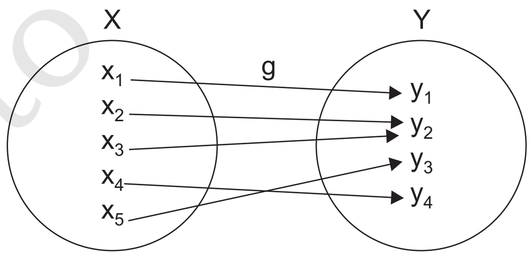
INTO FUNCTION: A function $f : A \rightarrow B$ is a into function if there exists an element in B having no pre-image in A.

In other words, $f : A \rightarrow B$ is an into function it it is not an onto function.

Illustration 1. *Let* $f : A \rightarrow B$ *is an into function, if it is not an onto function.*



(a)



(b)

Clearly, b_2 *and* b_3 *are two elements in B which do not have their pre-images in A. So,* $f : A \rightarrow B$ *is an into functions.*

Then, *f is onto because* $f(A) = \{f(-1), f(1, f(2), f(-2)\} = 0$, A = B.

14.5. BIJECTION (ONE-ONE ONTO FUNCTION)

Definition: A function $f : A \to B$ is a bijection if it is one-one as well as onto.

In other words, a function $f : A \to B$ is a bijection, if it is

(*i*) one-one *i.e.* $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$.

(*ii*) onto *i.e.* for all $y \in B$, there exists is such that $f(x) = y$.

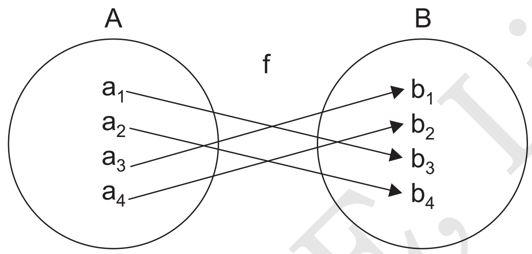


Illustration 1. *Let $f : A \to B$ be a function represented by the following diagram.*

Clearly, f is a bijection since it is both objective as well as surjective.

Illustration 2. *Prove that the function $f : Q \to Q$ given by $f(x) = 2x - 3$ for all $x \in Q$ is a bijection.*

Solution. Let x, y be two arbitrary elements in Q. Then,

$$f(x) = f(y) \Rightarrow 2x - 3 = 2y - 3 \Rightarrow 2x = 2y \Rightarrow x = y$$

$$f(x) = f(y) \Rightarrow x = y \text{ for all } x, y \in Q.$$

So, f is an injective map.

Surjectivity: Let y be an arbitrary element of Q. Then

$$f(x) = y \Rightarrow 2x - 3 = y \Rightarrow x = \frac{y-3}{2}$$

Clearly, for all $y \in Q$, $x = \frac{y+3}{2} \in Q$. Thus, for all $y \in Q$ (co-domain) there exists $x \in Q$ (domain) given by $x = \frac{y+3}{2}$ such that $f(x) = f\left(\frac{y+3}{2}\right) = 2\left(\frac{y+3}{2}\right) - 3 = y$. That is every element in the co-domain has its pre-image in x.

So, f is a surjection.

Hence, $f : Q \to Q$ is a bijection.

Ratio and Proportion

1. **Ratio:** The ratio of two quantitites a and b in the same units is the fraction $\frac{a}{b}$ and we write it $a : b$. In the ratio $a : b$, we call a as the first term or antecedent and b, the second term or consequence.

   Ex. The ratio 5 : 9 represents $\frac{5}{9}$ with antecedent = 4, consequent = 9

   Rule: The multiplication or division of each term of a ratio by the same non-zero number does affect the ratio.

   Ex. 4 : 5 = 8 : 10 = 12 : 15 etc. Also, 4 : 6 = 2 : 3

2. **Proportion:** The equality of two ratios is called proportion.

   If $a : b = c : d$, we write, $a : b : c : d$ and we say that a, b, c, d are in proportion.

   Here a and d are called extremes, while b and c are called mean terms.

   Produt of means = Product of extremes

   Thus, $a : b :: c : d \Rightarrow (b \times c) = (a \times d)$

3. (*i*) **Fourth proportional:** If $a : b = c : d$ then d is called the fourth proportional to a, b, c.

   (*ii*) **Third proportional:** If $a : b = c : d$ then c is called the third proportional to a, b, c.

   (*iii*) **Mean proportional:** Mean proportional between a and b is $\sqrt{ab}$

4. (*i*) **Comparison of Ratios:** We say that $(a : b) > (c : d) \Leftrightarrow \frac{a}{b} > \frac{c}{d}$

   (*ii*) **Compounded Ratio:** The Compounded ratio of the ratios $(a : b)$, $(c : d)$, $(e : f)$ is $(ace : bdf)$.

5. (*i*) Dublicate ratio of $(a : b)$ is $(a^2 : b^2)$

   (*ii*) Sub-dublicate ratio of $(a : b)$ is $(\sqrt{a} : \sqrt{b})$

   (*iii*) Triplicate ratio of $(a : b)$ is $(a^3 : b^3)$

   (*iv*) Sub-triplicate ratio of $(a : b)$ is $(a^{1/3} : b^{1/3})$

   (*v*) If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$, (componendo and dividendo)

6. Variation

(*i*) We say that x is directly proportional to y, if $x = ky$ for some constant k and we write, $x \propto ky$.

(*ii*) We say that x is inversely proportional to y, if $xy = k$ for some constant k and we write, $x \propto ky$.

Example 2. *If 10% of x is equal to 20% of y, then find x : y.*

Solution. 10% of x = 20% of y

$$\Rightarrow \quad \frac{10}{100}x = \frac{20}{100}y \Rightarrow \frac{x}{10} = \frac{y}{5} \Rightarrow \frac{x}{y} = \frac{10}{5} = \frac{2}{1}$$

Hence, $x : y = 2 : 1$.

Example 3. *A man spends ₹ 500 in buying 12 tables and chairs. The cose of one table is ₹ 50 and that of one chair is ₹ 40. What is the ratio of the numbers of the chairs and tables purchased?*

Solution. Let the number of tables purchased be x.

Then, number of chairs purchased be x.

$$\therefore \quad 50x + 40(12 - x) = 500$$

$$\Leftrightarrow \quad 50x - 40x = 500 - 480$$

$$\Leftrightarrow \quad 10x = 20$$

$$\Leftrightarrow \quad x = 2$$

So, number of tables = 2 and number of chairs = 10

Hence, requried ratio = 10 : 2 = 5 : 1.

Problems on Variation

In math variation we solved numerous types of problems on variation by using different types of variation like direct variation, inverse variation and joint variation. The problems on variation are mainly related to the questions based on word problems of constant variation, word problems of direct variation, word problems on inverse variation and also word problems of joint variation. Each word problems on variation are explained step by step so that students can understand the question and their solution easily.

Example 4. *The area of an umbrealla varies directly as the square of its radius. If the radius at the umbrella is doubled, how much will be the area of the umbrella?*

Solution. If the area of the umbrella is C and radius is R then $C = R^2$ or $C = KR^2$ where K is the constant of variation.

So, the area of the umbrella is KR^2.

Now if the radius is doubled the area will be

$$K(2R)^2 = 4KR^2 = 4C$$

So, the area will be by 4 times of normal the area of the umbrella.

Example 5. *The volume of a globe varies as the cube of its radius. Three solid globes of diameters 1½, 2 and 2½ metres are melted and formed into a new solid globe. Find the diameter of the new globe.*

Solution. $V \propto R^3$

Therefore $V = kR^3$ ...(1) [k = constant of variation]

If V_1, V_2 and V_3 cubic metres be the respective volumes of globes having radii ¾, 1 and 5/4 metres then using (1) we get,

$$V_1 = k.(3/4)^3 = 27k/64;$$
$$V_2 = k.1^3 = k;$$
$$V_3 = k.(5/4)^3 = 125k/64$$

Let V cubic metre be the volume of the new solid globe. Then,

$$V = V_1 + V_2 + V_3$$

or $$V = 27k/64 + k + 125k/64$$

or $$V = 216k/64$$

or $$V = 27k/8$$

Example 6. *If the radius of the new solid globe be r metre, then using* (*1*) *we get,*

Solution. $$V = kr_3$$

or $$kr^3 = 27k/8 \quad \text{or} \quad r^3 = (3/2)^3$$

or $$r = 3/2$$

Therefore, the diameter of the new globe = $2r = 2 \cdot 3/2 = 3$ metres.

EXERCISE

1. Let A = {x, y, z} and R = {1, 2}. Find the number of relations from A to B.
2. If $a : 5 = b : 7 = c : 8$, then $\dfrac{a+b+c}{a}$?
3. If $p : q = r : s = t : u = 2 : 3$, then find $(mp + nr + ot) : (mq + ns + ot)$.
4. In X is in indirect variation with square of Y and when X is 3, Y is 4. What is the value of X when Y is 4?

PERIOD-IV

M12CH15

CHAPTER

15

Algebraic Process

15.1. SIMPLIFICATION

Simplification is a process of replacing a mathematical expression by an equivalent one, that is simpler (usually shorter) for example. Simplification of algebraic expressions in computer algebra. The important fact and formulae given below:

I. 'BODMAS' Rule: This rule depicts the correct sequence in which the operations are to be executed, so as to find out the value of a given expression.

Here 'B' stands for 'Bracket', 'O' stands for 'of', 'D' stands for 'Division', 'M' stands for 'Multiplication', 'A' stands for 'Addition', and 'S' stands for 'Subtraction'.

Thus, in simplifying an expression, first of all the brackets must be removed, strictly in the order:

(*i*) Of (*ii*) Division (*iii*) Multiplication

(*iv*) Addition (*v*) Subtraction

II. Modulus of a Real Number: Modulus of a real number a is defined as

$$|a| = \begin{cases} a, & \text{if } a > 0 \\ -a, & \text{if } a < 0 \end{cases}$$

Thus, $|5| = 5$ and $|-5| = -(-5) = 5$.

III. Virnaculum (or Bar): When an expression contains Virnaculum, before applying the 'BODMAS' rule, we simplify the expression under the virnaculum.

IV. Some Important Formulae:

(*i*) $(a + b)^2 = (a^2 + b^2 + 2ab)$

(*ii*) $(a - b)^2 = (a^2 + b^2 - 2ab)$

(*iii*) $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$

(*iv*) $(a + b)^2 - (a - b)^2 = 4ab$

(*v*) $(a^2 - b)^2 = (a + b)\ (a - b)$

(*vi*) $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

(*vii*) $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$

(*viii*) $(a^3 + b^3) = (a + b)\ (a^2 - ab + b^2)$

(*ix*) $(a^3 - b^3) = (a - b)\ (a^2 + ab + b^2)$

(*x*) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

(*xi*) $a^3 + b^3 + c^3 - 3abc = (a + b + c)\ (a^2 + b^2 + c^2 - ab - bc - ca)$

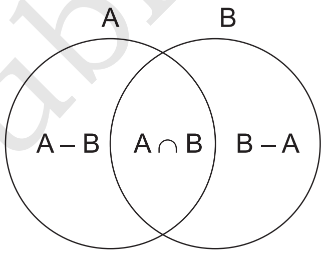
V. For any two sets A and B, we have:

(*i*) $n(\text{A} - \text{B}) + n(\text{A} \cap \text{B}) = n(\text{A})$

(*ii*) $n(\text{B} - \text{A}) + n(\text{A} \cap \text{B}) = n(\text{B})$

(*iii*) $n(\text{A} \cup \text{B}) = n(\text{A} - \text{B}) + n(\text{A} \cap \text{B}) + n(\text{B} - \text{A})$

(*iv*) $n(\text{A} \cup \text{B}) = n(\text{A}) + n(\text{B}) - n(\text{A} \cap \text{B})$



Example 1. *4368 + 2158 – 596 –? = 3421 + 1262*

Solution. Let $4368 + 2158 - 596 - x = 3421 + 1262$

$\Rightarrow \quad x + 596 = (4368 + 2158) - (3421 + 1262)$

$\Rightarrow \quad x + 596 = 6526 - 4683 = 1843$

$\Rightarrow \quad x = 1843 - 596 = 1247$

Hence, required number = 1247.

Example 2. *3456 ÷ 12 ÷ 8 = ?*

Solution. Given expression = $\dfrac{3456}{12} \div 8 = 288 \div 8 = 36.$

Example 3. *13 × 252 ÷ 42 + 170 = ? + 47*

Solution. Let $13 \times 252 \div 42 + 170 = x + 47$. Then

$$13 \times \frac{252}{42} + 170 = x + 47$$

$\Rightarrow \quad 13 \times 6 + 170 = x + 47$

$\Rightarrow \quad x + 47 = 78 + 170 = 248$

$\Rightarrow \quad x = 248 - 47 = 201.$

Hence required number = 201.

Factorization: Factorization can be defined as the process of breaking down a number into a smaller numbers which when multiplied together arrive at the original number. These numbers are broken down into factors or divisors. For example, 15 can be broken down as 3 × 5 and these two numbers are called factors.

Example 4. *Reduce the algebraic fractions to their lowest terms:* $\frac{x^2 - y^2}{x^3 - x^2y}$.

Solution. $\frac{x^2 - y^2}{x^3 - x^2y}$

Factorizing the numerator and denominator separately and cancelling the common factors, we get

$$= \frac{(x+y)(x-y)}{x^2(x-y)} = \frac{x+y}{x^2}$$

Example 5. *Simplify the algebraic fractions* $\frac{36x^2 - 4}{9x^2 + 6x + 1}$

Solution. $\frac{36x^2 - 4}{9x^2 + 6x + 1}$

Step 1: Factorize the numerator: $36x^2 - 4$

$$= 4(9x^2 - 1) = 4[(3x^2) - (1)^2]$$
$$= 4(3x + 1)(3x - 1)$$

Step 2: Factorize the denominator: $9x^2 + 6x + 1$

$$= 9x^2 + 3x + 3x + 1$$
$$= 3x(3x + 1) + 1(3x + 1)$$
$$= (3x + 1) + (3x + 1)$$

Step 3: Simplification of the given expression after factorizing the numerator and the denominator:

$$= \frac{36x^2 - 4}{9x^2 + 6x + 1} = \frac{4(3x+1)(3x-1)}{(3x+1)(3x+1)}$$
$$= \frac{4(3x-1)}{(3x+1)}$$

15.2. METHOD OF SUBSTITUTION

In this method, we express one of the variables in terms of the other from one of the two equations and then substituting the value of the variable to the other equation to get an equation in one variable.

Steps:

1. Choose any one of the two given equations and find the value of one variable (say *x*) in terms of the other (say *y*)

   For example:

   $$a_1x + b_1y + c_1 = 0 \qquad \ldots(1)$$

   $$\Rightarrow \quad a_1x = -c_1 - b_1y$$

   $$\therefore \quad x = \frac{-c_1 - b_1y}{a_1}$$

2. Substitute the value of *x*, obtained in step 1 to the second equation to get an equation in *y*

   Example:

   $$a_2x + b_2y + c_2 = 0 \qquad \ldots(2)$$

   $$a_2 \times \frac{(-c_1 - b_1y)}{a_1} + b_2y + c_2 = 0$$

   ∴ Equation (*ii*) reduces to an equation in *y*.

3. Solve the equation obtained in step 2 to get the value of *y*.
4. Substitute the value of *y* in equation (1) to get the value of *x*.
5. The values of *x* and *y* obtained in steps 3 and 4 are solution of pair of linear equations in two variables.

Example 6. *Solve the following pair of linear equations by substitution method:*

$$x + 2y = -1$$
$$2x + 3y = 12$$

Solution. (*i*) We have,

$$x + 2y = -1 \qquad \ldots(i)$$
$$2x + 3y = 12 \qquad \ldots(ii)$$

From (*i*), $x = -1, -2y \qquad \ldots(iii)$

Substituting $x = -1 - 2y$ in (*ii*), we get

$$2(-1 - 2y) + 3y = 12$$

$$\Rightarrow \quad -2 - 4y + 3y = 12$$

$$\Rightarrow \quad -y = 12 + 2 = 14$$

$$\therefore \quad y = -14$$

Substituting $y = -14$ in (*iii*), we get

$$x = -1 - 2x - 14$$

$$\therefore \quad x = 27$$

So, $x = 27$ and $y = -14$ Ans.

Example 7. *Solve the following pair of linear equation by the substitution method.*

$$\frac{x}{3} + \frac{y}{4} = 11$$

$$\frac{5x}{6} - \frac{y}{3} = -7$$

Solution.

$$\frac{x}{3} + \frac{y}{4} = 11 \qquad \ldots(i)$$

$$\frac{5x}{6} - \frac{y}{3} = -7 \qquad \ldots(ii)$$

$$\Rightarrow \quad \frac{x}{3} + \frac{y}{4} = 11$$

$$\Rightarrow \quad \frac{x}{3} = 11 - \frac{y}{4}$$

$$\Rightarrow \quad \frac{x}{3} = \frac{44 - y}{4}$$

$$\Rightarrow \quad x = \frac{3(44 - y)}{4} \qquad \ldots(iii)$$

Substituting $x = \dfrac{3(44 - y)}{4}$ in (*ii*), we get

$$\Rightarrow \quad \frac{5 \times 3(44 - y)}{6 \times 4} - \frac{y}{3} = -7$$

$$\Rightarrow \quad \frac{5}{8}(44 - y) - \frac{y}{3} = -7$$

$$\Rightarrow \quad \frac{15(44 - y) - 8y}{24} = -7$$

$\Rightarrow \quad 660 - 15y - 8y = -168$

$\Rightarrow \quad -23y = -168 - 660$

$\Rightarrow \quad 23y = 828$

$\Rightarrow \quad y = \frac{828}{23}$

$\therefore \quad y = 36$

Substituting $y = 36$ in (*iii*), we get

$$x = \frac{3(44-36)}{4}$$

$$\therefore \quad x = \frac{3 \times 8}{4} = 6$$

Hence solution of pair of equations is $x = 6$ and $y = 36$. **Ans.**

INEQUATION: A *statement involving variable(s) and the sign of inequality viz, >, <, ≥ or ≤ is called an inequation or an inequality.*

An inequation may contain one or more variables. Also, it may be linear or quadratic or cubic etc.

Following are some examples of inequations:

(*i*) $3x - 2 < 0$ (*ii*) $2x + 3 \le 0$

(*iii*) $5x - 3 > 0$ (*iv*) $4x + 5 \ge 0$

(*v*) $2x + 3y < 1$ (*vi*) $5x + 4y \le 3$

(*vii*) $4x - 6y > 5$ (*viii*) $2x + 5y \ge 4$

(*ix*) $2x^2 + 3x + 4 > 0$ (*x*) $x^2 - 3x + 2 \ge 0$

(*xi*) $x^2 + 3x + 2 < 0$ (*xii*) $x^2 - 5x + 4 \le 0$

(*xiii*) $x^3 - 6x^2 + 11x - 6 > 0$ (*xiv*) $x^3 + 6x^2 + 11x + 6 \le 0$

Linear Inequation in One Variable

Let a be a non-zero real number and x be a variable. Then inequations of the form ax + b < 0, ax + b ≤ 0, ax + b > 0 and ax + b ≥ 0 are known as linear inequations in one variable x.

For example, $9x - 15 > 0$, $5x - 4 \ge 0$, $3x + 2 < 0$ and $2x - 3 \le 0$ are linear inequations in one variable.

Linear Inequations in Two Variables

Let a, b be non-zero real numbers and x, y be variables. Then inequations of the form ax + by < c, ax + by ≤ c, ax + by > c and ax + by ≥ c are known as linear inequations in two variables x and y.

For example, $2x + 3y \leq 6$, $3x - 2y \geq 12$, $x + y < 4$, $2x + y \geq 6$ are linear inequations in two variables x and y.

Quadratic Inequation

Let a be A non-zero real number. Then an inequation of the form $ax^2 + bx + c < 0$, *or* $ax^2 + bx + c \leq 0$ *or* $ax^2 + bx + c > 0$ *or* $ax^2 + bx + c \geq 0$ *is known as a quadratic inequation.*

For example, $x^2 + x - 6 < 0$, $x^2 - 3x + 2 \geq 0$, $2x^2 + 3x + 1 > 0$ and $x^2 - 5x + 4 \leq 0$ are quadratic inequations.

15.3. SOLUTIONS OF AN INEQUATION

Definition: *A solution of an inequation is the value(s) of the variable(s) that makes it a true statements.*

Consider the inequation $\frac{3-2x}{5} < \frac{x}{3} - 4$.

Left hand side (LHS) of this inequation is $\frac{3-2x}{5}$ and right hand side (RHS) is $\frac{x}{3} - 4$

We observe that:

For $x = 0$, we have

$$\text{LHS} = \frac{3 - 2 \times 9}{5} = -3 \text{ and RHS} = \frac{9}{3} - 4 = -1$$

Clearly, $-3 < -1$

$\Rightarrow$ LHS < RHS, which is true

So, $x = 9$ is a solution of the given inequation.

For $x = 6$, we have

$$\text{LHS} = \frac{3 - 2 \times 6}{5} = -\frac{9}{5} \text{ and RHS} = \frac{6}{3} - 4 = -2$$

Because, $-\frac{9}{5} < -2$ is not true. So, $x = 6$ is not a solution of the given inequation.

We can verify that any real number greater than 7 is a solution of the given inequation.

Let us now consider the inequation $x^2 + 1 < 0$.

We know that

$$x^2 \geq 0 \text{ for all } x \in \mathrm{R}$$

$$\therefore \quad x^2 + 1 \geq 1 \text{ for all } x \in \mathrm{R}$$

$$\Rightarrow \quad x^2 + 1 \nless 0 \text{ for any } x \in \mathrm{R}$$

So, there is no real value of x which makes the given inequation a true statement. Hence, it has no solution.

It follows from the above discussion that an inequation may or may not have a solution. However, if an inequation has a solution it may have infinitely many solutions.

Solving an Inequation: *It is the process of obtaining all possible solutions of an inequation.*

Solution Set: *The set of all possible solutions of an inequation is known as its solution set.*

For example, the solution set of the inequation $x^2 + 1 \geq 0$ is the set R of all real numbesr where as the solution set of the inequation $x^2 + 1 < 0$ is the null set ϕ.

15.4. SOLVING LINEAR INEQUATIONS IN ONE VARIABLE

As mentioned in the previous section that solving an inequation is the process of obtaining its all possible solutions. In the process of the solving an inequation, we use mathematical simplifications which are governed by the following rules:

Rule 1: *Same number may be added to (or subtracted from) both sides of an inequation without changing the sign of inequality.*

Rule 2: *Both sides of an inequation can be multiplied (or divided) by the same positive real number without changing the sign of inequality. However, the sign of inequality is reversed when both sides of an inequation are multiplied or divided by a negative number.*

Rule 3: *Any term of an inequation may be taken to the other side with its sign changed without affecting the sign of inequality.*

A linear inequation in one variable is of the form.

$ax + b < 0$ or $ax + b \leq 0$ or, $ax + b > 0$ or, $ax + b \geq 0$

Example 8. *Solve the following linear inequations:*

(*i*) $2x - 4 \leq 0$ (*ii*) $-3x + 12 < 0$

(*iii*) $4x - 12 \geq 0$ (*iv*) $7x + 9 > 30$

Solution. (*i*) We have,

$$2x - 4 \leq 0$$

$$\Rightarrow \quad (2x - 4) + 4 \leq 0 + 4 \qquad \text{[Adding 4 on both sides]}$$

$$\Rightarrow \quad 2x \leq 4 \quad \Rightarrow \quad \frac{2x}{2} \leq \frac{4}{2} \quad \Rightarrow \quad x \leq 2$$

Hence, any real number less than or equal to 2 is a solution of the given inequation.

These solutions can be graphed on real line as shown in Figure.

The solution set of the given inequation is (– ∞. 2)

(*ii*) We have,

$$-3x + 12 < 0$$

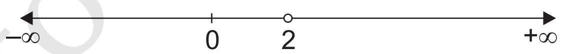
$$\Rightarrow \quad -3x < -12 \qquad \text{[Transposing 12 on right side]}$$

$$\Rightarrow \quad \frac{-3x}{-3} < \frac{-12}{-3} \qquad \text{[Dividing both sides by – 3]}$$

$$\Rightarrow \quad x > 4$$

Thus, any real number greater than 4 is a solution of the given inequation,

Hence, the solution set of the given inequation is $(4, \infty)$. This solution set can be graphed on real line as shown in Figure.



(*iii*) We have

$$4x - 12 \geq 0$$

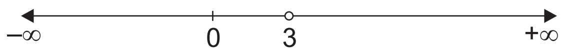
$$\Rightarrow \quad 4x \geq 12 \qquad \text{[Transposing 12 on RHS]}$$

$$\Rightarrow \quad \frac{4x}{4} \geq \frac{12}{4} \qquad \text{[Dividing both sides by 3]}$$

$$\Rightarrow \quad x \geq 3$$

$$\Rightarrow \quad x \in (3, \infty)$$

Hence, the solution set of the given inequation is $(3, \infty)$. This solution set can be graphed on real line as shown in Figure.



(*iv*) We have

$$7x + 9 > 30$$

$\Rightarrow \quad 7x > 30 - 9$ [Transposing 9 on RHS]

$\Rightarrow \quad 7x > 21$

$\Rightarrow \quad \dfrac{7x}{7} > \dfrac{21}{7}$

$\Rightarrow \quad x > 3$

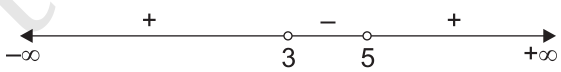
$\Rightarrow \quad x \in (3, \infty)$

Hence, $(3, \infty)$ is the solution set of the given inequation. This can be graphed on real line as shown in Figure

Example 9. *Solve the following linear inequations:* $\dfrac{x-3}{x-5} > 0$

Solution. We have, $\dfrac{x-3}{x-5} > 0$

Equating $x - 3$ and $x - 5$ to zero, we obtain $x = 3, 5$ as critical points. Plot these points on real line as shown in Figure. The real line is divided into three regions. In the right must region the expression on LHS of equation is positive and in the remaining two regions it is alternatively negative and positive as shown in Figure.



Since the expression in (*i*) is positive, so the solution set of the given inequation is the union of regions containing positive signs. Hence from Figure.

$$\frac{x-3}{x-5} > 0 \Rightarrow x \in (-\infty, 3) \cup (5, \infty)$$

Hence, the solution set of the given inequation is $(-\infty, 3) \cap (5, \infty)$ as shown in Figure.

15.5. SOLUTION OF SIMULTANEOUS LINEAR INEQUATIONS IN TWO VARIABLE

In this section, we will discuss the technique of finding the solution set of simultaneous linear equations. Solving simultaneous linear inequations means finding the set of points (x, y) in which all the constraints are satisfied. Note that the solution set of simultaneous linear inequations may be an empty set or it may be the region bounded by the straight lines corresponding to linear inequations or it may be an unbounded region with straight lines boundaries.

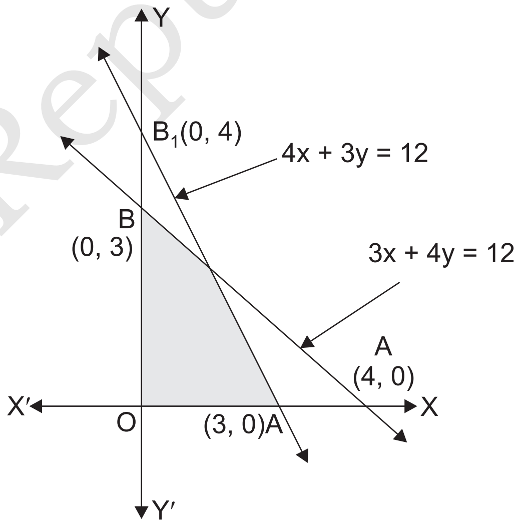
Example 10. *Exhibit graphically the solution set of the linear inequations.*

$$3x + 4y \leq 12,\ 4x + 3y \leq 12,\ x \geq 0,\ y \geq 0$$

Solution. Converting the inequations into equations, the inequations reduce to

$$3x + 4y = 12,\ 4x + 3y = 12,\ x = 0,\ y = 0$$

Region Represented by $3x + 4y \leq 12$: The line $3x + 4y = 12$ meets the coordinate axes at A(4, 0) and B(0, 3). Draw a thick line joining A and B. We find that (0, 0) satisfies inequation $3x + 4y \leq 12$. So, the portion containing the origin represents the solution set of the inequation $3x + 4y \leq 12$.



Region Represented by $4x + 3y \leq 12$: The line $4x + 3y = 12$ meets the x and y-axes at A_1(3, 0) and B_1(0, 4) respectively. Join these two points by a thick line. Clearly, the region containing the origin is represented by the inequation $4x + 3y \leq 12$.

Region Represented by $x \geq 0$ and $y \geq 0$: Clearly, $x \geq 0$ and $y \geq 0$ represent the first quadrant.

Hence, the shaded region given in Figure represents the solution set of the given linear inequations.

Example 11. *Exhibit graphically the solution set of the linear inequations.*

$$x + y \leq 5,\ 4x + y \geq 4,\ x + 5y \geq 5,\ x \leq 4,\ y \leq 3,$$

Solution. Converting the inequations into equations, we obtain

$$x + y = 5,\ 4x + y = 4,\ x + 5y = 5,\ x = 4,\ y = 3,$$

Region Represented by $x + y \leq 5$: The line $x + y = 5$ meets the coordinate axes at A(5, 0) and B(0, 5) respectively. Join these points by a thick line. Clearly (0, 0) satisfies the inequality $x + y \leq 5$. So, the portion containing the origin represents the solution set of the inequation $x + y \leq 5$.

Region Represented by $4x + y \geq 4$: The line $4x + y = 4$ meets the coordinate axes at A_1(1, 0) and B_1(0, 4) respectively. Join these points by a thick line. Clearly, (0, 0) does not satisfy the inequation $4x + y \geq 4$. So, the portion not containing the origin is represented by the inequation $4x + y \geq 4$.

Region Represented by $x + 5y \geq 5$: The line $x + 5y \geq 5$ meets the coordinate axes at A(5, 0) and B_2(0, 1) respectively. Join these two points by a thick line. We find that (0, 0) does not satisfy the inequation $x + 5y \geq 5$. So, the portion not containing the origin is represented by the given inequation.

Region Represented by $x \leq 4$: Clearly, $x = 4$ is a parallel to y-axis at a distance of 4 units from the origin. Since (0, 0) satisfies the inequation $x \leq 4$. So, the portion lying on the left side of $x = 4$ is the region represented by $x \leq 4$.

Region Represented by $y \leq 3$: Clearly, $y = 3$ is a line parallel to y-axis at a distance 3 from it. Since (0, 0) satisfies $y \leq 3$. So, the portion containing the origin is represented by the given inequation.

The common region of the above five regions represents the solution set of the given linear constants as shown in Figure.

Simultaneous Linear Equations: A set of two or more equations each containing two or more variables whose values can simultaneously satisfy both or all the equations in the set, the numbers of variables being equal to or less than the number of equations in the set.

15.6 SOLVING SIMULTANEOUS LINEAR EQUATIONS

(*i*) Graphical method

1. Obtain the pair of linear equations

$$a_1x + b_1y + c_1 = 0 \quad \ldots(1)$$

and $$a_2x + b_2y + c_2 = 0 \quad \ldots(2)$$

2. Find atleast two solution for each of the two equations by assuming value of one variable and then calculating the other variable.

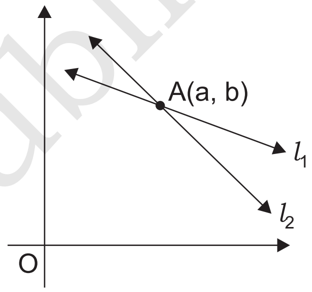
3. Plot these points (solutions) of both the equation in the same co-ordinate axes to get two straight line, one for each equation.

While plotting the graph, the following three cases arises:

Case I: The two lines intersect at a point A (Figure)

Then the two equations have unique solutions given by $x = a$ and $y = b$.

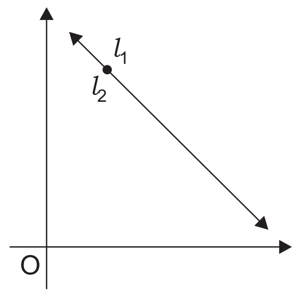
The equations are said to be consistent.



Case II: The two lines coincide each other (Figure)

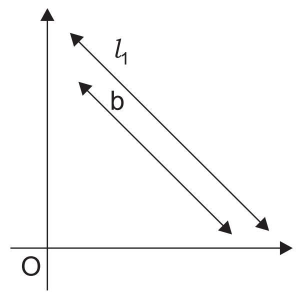
Then the two equations have infinitely many solutions.

The equations are said to be consistent.



Case III: The two lines are parallel to each other.

Then the two equations have no solutions and are said to be inconsistent.



Example 12. *Solve the following pair of equation graphically*

$$x + y = 3$$

$$2x + 5y = 12$$

Solution. We have,

$$x + y = 3 \qquad \ldots(1)$$

If $x = 1$, $y = 2$

If $x = 2$, $y = 1$

Two solutions are

x	1	2
y	2	1

and $2x + 5y = 12$

If $x = 1$, $2 \times 1 + 5y = 12 \Rightarrow 5y = 10$

$\therefore$ $y = 2$

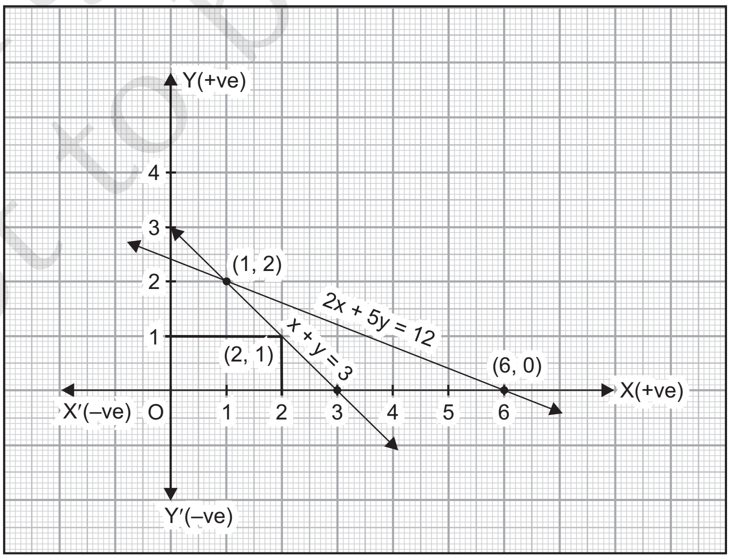
$x = 6$, $2 \times 6 + 5y = 12 \Rightarrow 5y = 0$

$\therefore$ $y = 0$

Two solutions are

x	1	6
y	2	0

Figure shows the graph of the equations. The two lines intersect at the point (1, 2).



15.7. QUADRATIC EQUATION

Any equation of the form $p(x) = 0$, where $p(x)$ is a polynomial of degree 2, is known as quadratic equation.

For Example: $2x^2 + 5x + 3 = 0$ is a quadratic equation.

An equation of the form $ax^2 + bx + c = 0$ where a, b, c are real number and $a \neq 0$ is known as the standard form of a quadratic equation.

15.8. ROOTS OF A QUADRATIC EQUATION

A real number α is a root of quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ if $a\alpha^2 + b\alpha + c = 0$. So any real number which satisfies a given quadratic equation is called the root of the quadratic equation.

In polynomial chapter, we learnt that if $x = \alpha$ satisfies a quadratic polynomial $p(x)$ *i.e.*, $p(\alpha) = 0$, then α is the zero of $p(x)$.

This means that zeroes of a quadratic polynomial $ax^2 + bx + c$ and roots of a quadratic equation $ax^2 + bx + c = 0$ are the same. A quadratic equation can have atmost two real roots.

15.9 SOLUTIONS OF A QUADRATIC EQUATION

Solving a quadratic equations means finding the roots of the quadratic equation.

Example 13. *Which of the following are quadratic equations?*

(*i*) $x^2 + 5x - 6 = 0$ (*ii*) $2x^2 + 6x + 8 = 0$

(*iii*) $x^2 - 5 = 0$ (*iv*) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$

Solution. (*i*) As $x^2 + 5x - 6$ is a quadratic polynomial

$\therefore$ $x^2 + 5x - 6 = 0$ is a quadratic equation

(*ii*) $(2x^2 + 6x + 8)$ is a quadratic polynomial

$\therefore$ $2x^2 + 6x + 8 = 0$ is a quadratic equation.

(*iii*) $(x^2 - 5)$ is a quadratic polynomial

$\therefore$ $x^2 - 5 = 0$ is a quadratic polynomial.

(*iv*) $\sqrt{2}x^2 + 7x + 5\sqrt{2}$ is a quadratic polynomial

$\therefore$ $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$ is a quadratic equation.

Linear Programming

Linear Programming is a method of finding an optimal value (i.e., maximum or minimum value) of a linear function of several variables subject to the conditions that variables are non-negative and satisfy a set of linear equations or inequations.

The linear function, which has to be maximised or minimised, is called objective function.

The process of maximisation or minimisation is called optimisation.

The variables involved in linear programming are called decision variables.

The restrictions on the decision variables to be non-negative are called non-negativity restrictions.

The restrictions on the decision variables to satisfy linear equations are called constraints.

In this chapter, we shall restrict our study to Linear Programming Problem in two variables and up to three non-trivial constraints. Also, we shall use the abbreviation L.P.P for the term 'Linear Programming Problem

A general L.P.P in two variables and up to three constraints is of the form

Maximise (or Minimise) $Z = ax + by\}$ Objective function

Subject to the constraints

$$\left.\begin{aligned} a_1x + b_1y \{=, >, \geq, <, \leq\} c_1 \\ a_2x + b_2y \{=, >, \geq, <, \leq\} c_2 \\ a_3x + b_3y \{=, >, \geq, <, \leq\} c_3 \end{aligned}\right\} \text{Constraints}$$

$$x \geq 0,\ y \geq 0\} \quad \text{Non-negativity restrictions}$$

In Class XI, we have discussed the graphical method of solving system of linear inequalities in two variables. This method plays an important role in the study of Linear Programming Problems. Before proceeding further, let us recall this graphical method.

15.10. GRAPHICAL METHOD OF SOLVING L.P.P.

In this section, we shall discuss a graphical method of solving L.P.P. involving two decision variables x and y. This graphical method is known as **'Corner Point Method'**

We will first defined the following important terms used in this method.

1. **Feasible Region:** The common region determined by the constraints and non-negativity restrictions of L.P.P is called *feasible region.*
2. **Feasible Solution:** A set of values of decision variables of L.P.P. satisfying the constraints and the non-negativity restrictions is called *feasible solution.* Every point in the feasible region is a feasible solution of the given L.P.P.
3. **Optimal Feasible Solution/Optimal Solution:** A feasible solution of L.P.P is said to be *optimal feasible solution* (or *optimal solution*) if it optimises (*i.e.,* maximises or minimises) the objective function.
4. **Bounded Feasible Region:** A feasible region is said to be *bounded,* if it can be enclosed within a circle.
5. **Unbounded Feasible Region:** A feasible region is said to be *unbounded,* if it cannot be enclosed within any circle *i.e.,* if it extends indefinitely in any direction.
6. **Infeasible Region:** The region other than the feasible region is called *infeasible region.*
7. **Infeasible Solution:** Any point outside the feasible region is called *infeasible solution* of the given L.P.P.

The following theorems are fundamental in solving the L.P.P.

Theorem 1: Let R be the feasible region for L.P.P and let Z = $ax + by$ be the objective function. When Z has an optimum value (*i.e.,* maximum or minimum value), where the variables x and y are subject to the constraints described by the linear inequalities, this optimal value must occur at a corner point of the feasible region.

Theorem 2: Let R be the feasible region for L.P.P and let Z = $ax + by$ be the objective function. If R is bounded, then the objective function Z has both maximum or minimum value on R and each of these occurs at a corner point of R. If the feasible region is unbounded, then a maximum or a minimum may not exist. However, if it exists, it must occur at a corner point of R.

(The proofs of these theorems are beyond the scope of this book).

15.11. CORNER POINT METHOD

The Corner Point Method for solving L.P.P., involving two decision variables x and y, consists of the following steps:

Step I. Find the feasible region of the given L.P.P, and determine its corner points.

Step II. Evaluate the objective function $Z = ax + by$ at each corner point. Let M and m be the largest and smallest values of Z at these corner points respectively.

Step III. If the feasible region is bounded, then M and m are maximum and minimum values of Z. If the feasible region is unbounded, then

(*i*) If the open half plane determined by $ax + by > M$ has no point in common with the feasible region, then M is maximum value of Z, otherwise Z has no maximum value.

(*ii*) If the open half plane determined by $ax + by < m$ has no point in common with the feasible region, then m is minimum value of Z, otherwise Z has no minimum value.

Remark: If Z has optimum (*i.e.*, maximum or minimum) value at any two corner points, then it has optimum value at all the points of the line segment joining those points.

Let us consider the following examples.

Example 14. *Maximise* $Z = 3x + 4y$, *subject to the constraints:*

$$x + y \leq 4, \qquad x \geq 0, \qquad y \geq 0$$

Solution. Given L.P.P is Maximise

$$Z = 3x + 4y$$

subject to the constraints

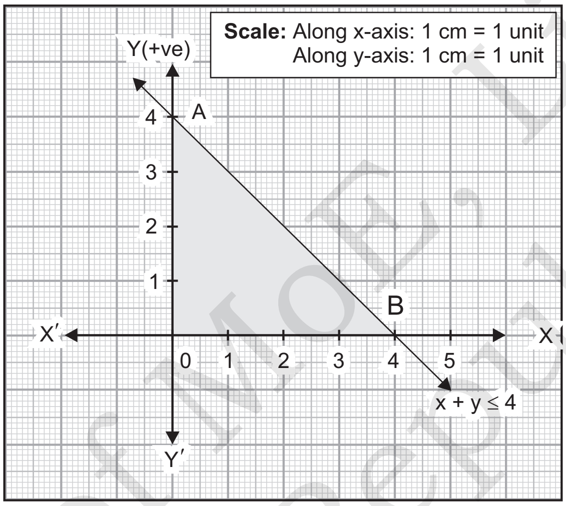
$$x + y \leq 4, \quad x \geq 0, \quad y \geq 0$$

consider the following equations:

$x + y = 4$ | $x = 0, \quad y = 0$

x	0	4
y	4	0

The feasible region of L.P.P. is bounded, as shown shaded in the graph.



Corner points	Value of Z(Z = 3x + 4y)
A(0, 4)	16
B(4, 0)	12
O(0, 0)	0

Since, the feasible region is bounded and 16 is the maximum value of Z at corner points.

Hence, 16 is the maximum value of Z in the feasible region at $x = 0$, $y = 4$.

Example 15. *Maximise Z = 8x + 9y subject to the constraints:*

$$2x + 3y \leq 6, \quad 3x - 2y \geq 6, \quad y \leq 1, x \geq 0, y \geq 0.$$

Solution. Given L.P.P is

Maximise $\quad Z = 8x + 9y$

Subject to the constraints:

$$2x + 3y \leq 6, \quad 3x - 2y \leq 6, \quad y \leq 1, x \geq 0, y \geq 0.$$

We consider the following equation

$2x + 3y = 6$

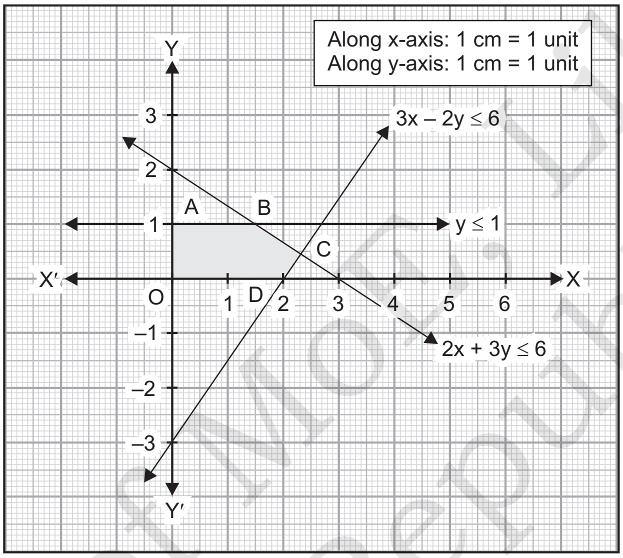
x	0	3
y	2	0

$3x - 2y = 6$

x	0	2
y	-3	0

$y = 1,\ x = 0,\ y = 0$

The feasible region of L.P.P. is bounded, as shown shaded in the graph.



Corner points	Value of Z(Z = $8x + 9y$)
A(0, 1)	9
$B\left(\frac{3}{2}, 1\right)$	21
$C\left(\frac{30}{13}, \frac{6}{13}\right)$	$\frac{294}{13}$
D(2, 0)	16
O(0, 0)	0

Since, the feasible region is bounded and $\frac{294}{13}$ is the maximum value of Z at corner points.

Hence, $\frac{294}{13}$ is the maximum value of Z in the feasible region at $x = \frac{30}{13}$, $y = \frac{6}{13}$.

15.12. WORD PROBLEMS ON LINEAR EQUATIONS

Worked-out word problems on linear equations with solutions explained step-by-step in different types of examples.

There are several problems which involve relations among known and unknown numbers and can be put in the form of equations. The equations are generally stated in words and it is for this reason we refer to these problems as word problems. With the help of equations in one variable, we have already practiced equations to solve some real life problems.

Steps involved in solving a linear equation word problem:

- Read the problem carefully and note what it is given and what is required and what is given.
- Denote the unknown by the variables x, y
- Translate the problem to the language of mathematics or mathematical statements.
- Form the linear equation in one variable using the conditions given in the problems.
- Solve the equation for the unknown.
- Verify to be sure whether the answer satisfies the conditions of the problem.

Example 16. *The length of a rectangle is twice its breadth. If the perimeter is 72 metre, find the length and breadth of the rectangle.*

Solution. Let the breadth of the rectangle be x,

Then the length of the rectangle = $2x$

Perimeter of the rectangle = 72

Therefore, according to the question

$$2(x + 2x) = 72 \quad \Rightarrow \quad 2 \times 3x = 72$$

$$\Rightarrow \quad 6x = 72 \quad \Rightarrow \quad x = 72/6$$

$$\Rightarrow \quad x = 12$$

We know, length of the rectangle = $2x$

$$= 2 \times 12 = 24$$

Therefore, length of the rectangle is 24 m and breadth of the rectangle is 12 m.

EXERCISE

1. (*a*) Simplify: (*a*) 460 × 15 – 5 × 20

(*b*) 1 ÷ (1 + 1 ÷ (1 + 1 ÷ 1(1 + 1 ÷ 2))) + 1

2. Find the missing numeral:

(*a*) (? – 2763) ÷ 86 × 13 = 208

(*b*) 3565 ÷ 23 + 4675 ÷ ? = 430

3. Simplify:

(*a*) $\dfrac{(6+6+6+6)\div 6}{4+4+4+4\div 4}$ (*b*) $\dfrac{(2+3)\times 5+3\div \frac{1}{2}}{6+5\times 4\div \frac{4}{5}}$

4. What should come in place of both the question marks in the following equation?

$$\frac{128\div 16\times ?-7\times 2}{7^2-8\times 6+?^2}=1$$

5. Factorise $4x^2 + 12x + 5$.

6. Factorise $y^2 + 16y + 60$.

7. Factorise $5x^2 + 14x - 3$.

PERIOD–V

M12CH16

TOPIC

16 Vector and Trigonometry

16.1. ORDERED PAIR

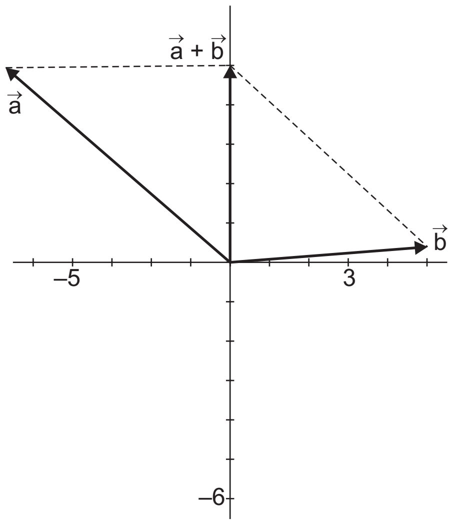
The ordered pair that describes the changes is $(x_2 - x_1, y_2 - y_1)$, in our examples (2 – 0, 5 – 0) or (2, 5). Two vectors are equal if they have the same magnitude and direction. They are parallel if they have the same or opposite direction. We can combine vectors by adding them, the sum of two vectors is called the resultant.

16.2. VECTORS

Vectors are used to represent a quantity that has both a magnitude and a direction. The vector is normally visualized in a graph. A vector between A and B is written as $\overrightarrow{AB}$

The vectors standard position has its starting point in origin.

The component form of a vector is the ordered pair that describes the changes in the x- and y-values. In the graph $x_1 = 0$, $y_1 = 0$, and $x_2 = 2$, $y_2 = 5$. The ordered pair that describes the changes is $(x_2 - x_1, y_2 - y_1)$, in our example (2 – 0, 5 – 0) or (2, 5).



Two vectors are equal if they have the same magnitude and direction. They are parallel if they have the same or opposite direction.

We can combine vectors by adding them, the sum of two vectors is called the resultant.

Example 1. *Add the two following vectors:*

$$\vec{a} = (2, 4), \ \vec{b} = (-1, 6)$$

Solution. We add the corresponding components

$$\vec{a} + \vec{b} = (2 + (-1), 4 + 6) = (1, 10)$$

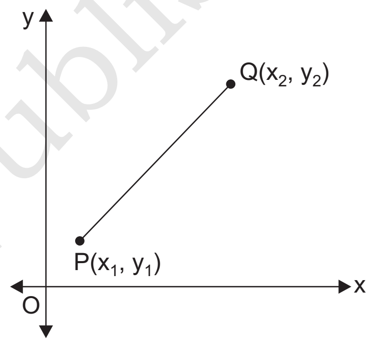
16.3. MAGNITUDE AND DIRECTION OF VECTORS

Magnitude of a Vector

The magnitude of a vector $\overrightarrow{PQ}$ is the distance between the initial point P and the end point Q. In symbols the magnitude of $\overrightarrow{PQ}$ is written as $|\overrightarrow{PQ}|$

If the coordinates of the initial point and the end point of a vector is given, the distance formula can be used to find its magnitude.

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Example 2. *Find the magnitude of the vector $\overrightarrow{PQ}$ whose initial point P is at (1, 1) and end point is at Q is at (5, 3).*

Solution. Use the Distance Formula

$$|\overrightarrow{PQ}| = \sqrt{(5-1)^2 + (3-1)^2} = \sqrt{4^2 + 2^2} = \sqrt{16 + 4}$$

$$= \sqrt{20} \approx 4.5$$

The magnitude of $\overrightarrow{PQ}$ is about 4.5

16.4. WHAT IS THE VECTOR ADDITION

Vectors are represented as a combination of direction and magnitude and are written with an alphabet and an arrow over them (or) with an alphabet written in bold. Two vectors, ***a*** and ***b***, can be added together using **vector addition**, and the resultant vector can be written as: ***a*** + ***b***.

Before learning about the properties of vector addition, we need to know about the conditions that are to be followed while adding vectors. The conditions are as follows:

- Vectors can be added only if they are of the same nature. For instance, acceleration should be added with only acceleration and not mass.
- We cannot add vectors and scalars together

Consider two vectors **C** and **D.** Where $\mathbf{C} = C_x i + C_y i + C_z k$ and $\mathbf{D} = C_x i + D_y j + D_z k$. Then, the resultant vector (or vector sum) $\mathbf{R} = \mathbf{C} + \mathbf{D} = (C_x + D_x)i + (C_y + D_y)j + (C_z + C_z)k$

Example 3. *Find the addition of vectors* $\overrightarrow{PQ}$ *and* $\overrightarrow{QR}$, *where PQ = (3, 2) and QR = (2, 6).*

Solution. We perform the vector addition by adding their corresponding components

$$PQ + QR = (3, 4) + (2, 6)$$
$$= (3 + 2, 4 + 6) = (5, 10)$$

16.5. WHAT IS THE VECTOR SUBTRACTION

The vector subtraction of two vectors ***a*** and ***b*** is represented by ***a* – *b*** and it is nothing but adding the negative of vector ***b*** to the vector ***a***. *i.e.* ***a* – *b*** = ***a* + (– *b*)**. Thus subtraction of vectors involves the addition of vectors and the negative of a vector. The result of vector subtraction is again a vector. The following are the rules for subtracting vectors

- It should be performed between two vectors only (not between one vector and scalar).
- Both vectors in the subtraction should represent the same physical quantity.

Example 4. *Compute the vector subtraction* $a - b$ *if* $a = < 1, -2, 5 >$ *and* $b = < 3, -1, 2 >$. *Also, find its magnitude.*

Solution. Given that $a = < 1, -2, 5 >$ and $b = < 3, -1, 2 >$

Now we will find their difference by subtracting the respective components.

$$a - b = < 1, -2, 5 > - < 3, -1, 2 >$$
$$= < 1 - 3, -2 - (-1), 5 - 2 >$$
$$= < -2, -1, 3 >$$

Its magnitude is

$$|a - b| = \sqrt{[(-2)^2 + (-1)^2 + 3^2]}$$
$$= \sqrt{(4 + 1 + 9)} = \sqrt{14}$$

16.6. SCALAR MULTIPLICATION OF VECTORS

To multiply a vector by a scalar, multiply each component by the scalar.

If $\vec{u} = (u_1, u_2)$ has magnitude $|\vec{u}|$ and direction d, then $n\vec{u} = n(u_1, u_2) = (n\,u_1, n\,u_2)$ where n is a positive real number, the magnitude is $|n\vec{u}|$, and its direction is d.

Not that if n is negative then the direction of nu is the opposite of d.

Example 5. *Let* $u = (-1, 3)$. *Find* $7u$.

Solution. $7u = 7(-1, 3)$

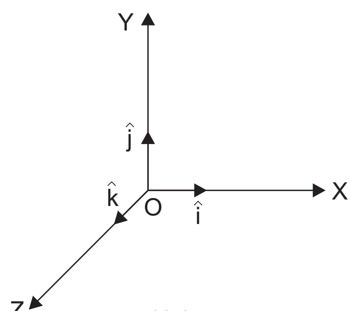
$$= (7(-1), 7(3))$$
$$= (-7, 21)$$

16.7. UNIT VECTOR SYMBOL

Unit Vector is represented by the symbol '^' which is called a cap or hat, such as: $\hat{a}$ it is given by $\hat{a} = \dfrac{a}{|a|}$

Where $|a|$ is for norm or magnitude of vector a.

It can be calculated using a Unit vector formula or by using a calculator.



Unit vector

16.8. UNIT VECTOR FORMULA

As explained above vectors have both magnitude (Value) and direction. They are shown with an arrow.

i.e., $\vec{a}$

And, $\hat{a}$ denotes a unit vector. If we want to change any vector in unit vector, divide it by the vector's magnitude.

Usually, *xyz* coordinates are used to write any vector.

It can be done in two ways:

1. $\vec{a} = (x, y, z)$ using the brackets.
2. $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$

Formula for magnitude of a vector is:

$$|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$$

Unit Vector = Vector/Vector's Magnitude

Unit Vector Example

Here is an example based on the unit vector. Observe and follow each step and solve problems based on it.

Example 6. *Find the unit vector $\vec{p}$ for the given vector, $12\hat{i} - 3\hat{j} - 4\hat{k}$ Show it in both formats – Bracket and Unit vector component.*

Solution. Let's find the magnitude of the given vector first,

$$|p| = \sqrt{x^2 + y^2 + z^2} = \sqrt{144 + 9 + 16} = \sqrt{169} = 13$$

Let's use this magnitude to find the unit vector now:

$$\hat{p} = \frac{p}{|p|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

$$\hat{p} = \frac{12\hat{i} - 3\hat{j} - 4\hat{j}}{13}$$

$$\hat{p} = \frac{12}{13}\hat{i} - \frac{3}{13}\hat{j} - \frac{4}{13}\hat{k}$$

The unit vector in Bracket form is:

$$\hat{p} = \left(\frac{12, -3, -4}{13}\right) = \left(\frac{(12)}{13}, -\frac{(3)}{13}, \frac{(-4)}{13}\right)$$

16.9. POSITION VECTOR

$$(\vec{r}) = x\hat{i} + y\hat{j} + z\hat{k}$$

Where,

$\hat{i}$ = unit vector along *x*-direction

$\hat{j}$ = unit vector along *y*-direction

$\hat{k}$ = unit vector along *z*-direction

16.10. POSITION VECTOR FORMULA

To find the position vector of any a point in the *xy*-plane, we should first know the point coordinates. Consider two points A and B whose coordinates are (x_1, y_1) and (x_2, y_2) respectively. To determine the position vector, we need to subtract the corresponding components of A from B as follows:

$$AB = (x_2 - x_1)i + (y_2 - y_1)j$$

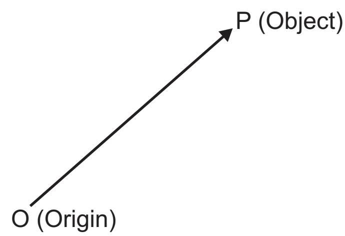
The position vector AB originates from point A and terminates at point B.

16.11. POSITION VECTOR DEFINITION

A position vector is defined as a vector that symbolises either the position or the location of any given point with respect to any arbitrary reference point like the origin. The direction of the position vector always points from the origin of that vector towards a given point.

16.12. POSITIVE VECTOR EXAMPLE

The position vector of an object is measured from the origin, in general. Suppose an object is placed in the space as shown in given figure.



16.13. DIRECTION OF VECTOR

The direction of a vector is the measure of the angle it makes with a horizontal line.

One of the following formulas can be use to find the direction of a vector.

$\tan\theta = \dfrac{y}{x}$, where x is the horizontal change and y is the vertical change.

or $\tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1}$, where (x_1, y_1) is the initial point and (x_2, y_2) is the terminal point.

Example 7. *Find the direction of the vector $\overrightarrow{PQ}$ whose initial point P is at (2, 3) and end point is at Q is at (5, 8).*

Solution. The coordinates of the initial point and the terminal point are given. Substitute them in the formula $\tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$

$$\tan\theta = \frac{8-3}{5-2} = \frac{5}{3}$$

Find the inverse tan, then use a calculator.

$$\theta = \tan^{-1}\left(\frac{5}{3}\right) \approx 59$$

16.14. WHAT IS STATIC EQUILIBRIUM?

A massive frame hung on a wall using two cables is in static equilibrium. A horizontal beam supported by a strut is in static equilibrium. So what is the definition of static equilibrium, and when do objects fall under this category?

Static equilibrium occurs when an object or a system remains at rest and does not tilt nor rotate. The word "static" means that the body is not in motion, while the term "equilibrium" indicate that all opposing forces are balanced. Thus, a system is in static equilibrium if it is at rest and all forces and other factors influencing the object are balanced.

16.15. STATIC EQUILIBRIUM EXAMPLES

Static equilibrium can be commonly observed in everyday life. Objects at rest are considered systems in static equilibrium, where both net force and net torque are zero. Two examples that demonstrate objects in static equilibrium are:

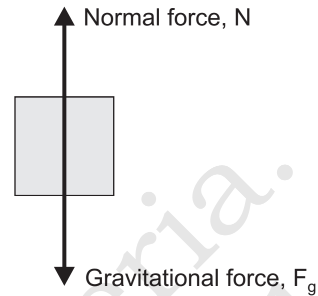
- a book placed on top of a table
- a balanced seesaw

Static Equilibrium

A book placed on top of a table is considered to be in static equilibrium. A free-body diagram, or a diagram showing all the forces acting on the object, can be used to check whether the object satisfies the two conditions of equilibrium.

A book at rest on top of the table is acted upon by the gravitational force and the normal force.

As shown in the diagram, the only forces acting on the book are gravity and normal force. Gravity acts downward, while normal force acts upward, perpendicular to the surface. The forces are also equal in magnitude but opposite in direction. Thus, the net force acting on the book is zero. There is also zero torque acting on the book, satisfying both the first and second conditions of static equilibrium.



16.16. PARALLEL VECTORS

The **parallel vectors** are vectors that have the same direction exactly the opposite direction i.e., for any vector *a*, the vector itself and its opposite vector – *a* are vectors that are always parallel to *a*. Extending this further, any scalar multiple of a is parallel to *a* i.e., a vector a and *ka* are always parallel vectors where '*k*' is scalar (real number).

Example 8. *Find the parallel vector if* $a = 2\hat{i} + 4\hat{j}$ *and* $b = 6\hat{i} + 12\hat{j}$.

Solution. We know that

$$\lambda a = b$$

$$\lambda\left(\frac{2}{4}\right) = \begin{pmatrix}6\\13\end{pmatrix}$$

$$2\lambda = 6 \quad \Rightarrow \quad \lambda = 3$$

$$4\lambda = 12 \quad \Rightarrow \quad \lambda = 3$$

λ are same, then vector are parallel.

16.17. WHAT IS AN ORTHOGONAL VECTOR?

In mathematical terms, the word orthogonal means directed at an angle of 90°. Two vectors *u, v* are orthogonal if they are perpendicular, i.e., they form a right angle, or if the dot product they yield is zero.

Example 9. *Find the* orthogonal *vector if A* = (*3, 4, 0*) *and B* = (– *4, 3 ,2*).

Solution. Two vectors A and B are orthogonal, if their dot product is zero. i.e., $A \cdot B = 0$

$$A \cdot B = A_1 \cdot B_1 + A_2 \cdot B_2 + A_3 \cdot B_3$$
$$= (3) \cdot (-4) + 4 \cdot 3 + 0 \cdot 2$$
$$= -12 + 12 = 0$$

So, vector are orthogonal.

16.18. LATITUDES AND LONGITUDES

Latitudes

Equator

- Equator is an imaginary line on the globe that divides it into two equal parts.
- The northern half of the earth is known as the Northern Hemisphere and Southern half is known as the Southern Hemisphere.

Parallels of Latitudes

- Parallels of latitudes are parallel circles from the equator up to the poles.
- They are measured in degrees.

The equator represents the zero degrees latitude. Its distance from the equator to either of the poles is one-fourth of a circle round the earth, it will measure 1/4th of 360 degrees, i.e., 90°. Thus, 90° degrees north latitude marks the North Pole and 90 degrees south latitude marks the South pole.

Latitudes and Longitudes (UPSC Notes)

Important Parallels and Latitudes

- Tropic of Cancer(23½°N) in the Northern Hemisphere.
- Tropic of Capricorn (23½°S) in the Souther Hemisphere.
- Arctic Circle at 66½° north of the equator.
- Antarctic Circle at 66½° south of the equator.

Longitudes

- The meridian which passed through Greenwich, where the British Royal Observatory is located. This meridian is considered as the Prime Meridian.
- Its values is 0° longitude and from it, we count 180° eastward as well as 180° westward. The Prime Meridian and 180° meridian divide the earth into two halves, the Eastern Hemisphere and the Western Hemisphere.

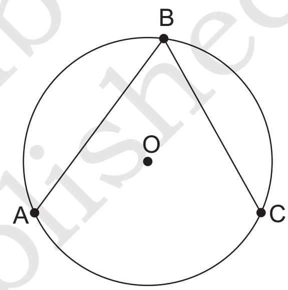
Longitude and Time

- The best means of measuring time is by the movement of the earth, the moon, and the planets. The sun regularly rises and sets every day.

- When the prime meridian of Greenwich has the sum at the highest point in the sky, all the places along this meridian will have mid-day or noon.
- As the earth rotates from west to east, those places east of Greenwich will be ahead of Greenwich time and those to the west will be behind it.

16.19. INSCRIBED ANGLES

An inscribe angle in a circle is formed by two chords that have a common end point on the circle. This common end point is the vertex of the angle.

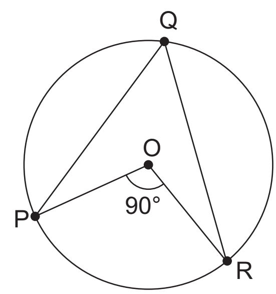


Here, the circle with center O has the inscribed angle ∠ABC. The other end points than the vertex, A and C defined the intercepted arc $\overset{\frown}{AC}$ of the circle. The measure of $\overset{\frown}{AC}$ is the measure of its central angle. That is, the measure of ∠AOC.

Example 10. *Find the measure of the inscribed angle ∠PQR.*

Solution. By the inscribed angle theorem, the measure of an inscribed angle is half the measure of the intercepted arc.

The measure of the central angle ∠POR of the intercepter arc $\overset{\frown}{PR}$ is 90°.



Therefore, $$m\angle PQR = \frac{1}{2} m\angle POR$$

$$= \frac{1}{2}(90°) = 45°$$

EXERCISE

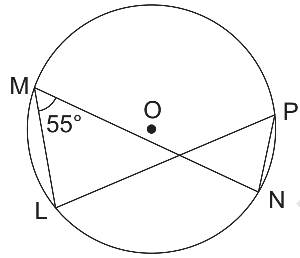
1. Find the magnitude of the vector $\overrightarrow{PQ}$ whose initial point P is at (2, 2) and end point is at Q is at (6, 8).
2. Find the addition of vectors PQ and QR, where PQ = (4, 3) and QR = (3, 5).
3. Compute the vector subtraction $a - b$ if $a = (1, -2, 5)$ and $b = (3, -1, 2)$.
4. Let $u = (-2, 6)$, find $3u$.
5. Find the unit vector $\vec{q}$ for the given vector, $-2\hat{i} + 4\hat{j} - 4\hat{k}$.
6. Find the unit vector $\vec{q}$ for the given vector, $-3\hat{i} + 9\hat{j} - 4\hat{k}$.

7. Find the direction of the vector $\overrightarrow{PQ}$ whose initial at (3, 4) and end point is at Q is at (6, 9).

8. Find the parallel vector if $a = 4\hat{i} + 8\hat{j}$ and $b = 12\hat{i} + 24\hat{j}$.

9. Find the orthogonal vector if A(4, 5, 0) and B = (– 5, 4, 0).

10. Find $m\angle$LPN.



PERIOD–V

M12CH17

TOPIC

17

Transformations and Coordinates

17.1. TRANSFORMATIONS

When a plane figure undergoes a **transformation** it means its position or shape or size has changed.

The following types of transformation shall be considered in this chapter:

1. Reflection
2. Rotation
3. Translation
4. Enlargement
5. Special mappings

Note: Reflection, rotation and translation are called ***rigid motions*** because under such transformations the shape or size of the figure transformed does not change.

Enlargement is not a rigid because the size of the figure transformed is changed,

Note that the transformation of plane figures can be done in the Cartesian plane by transforming the *vertices* of the plane figure.

17.2. REFLECTION

A reflection is the image you see when you look in a mirror. The mirror forms the line of symmetry between between the object and the image.

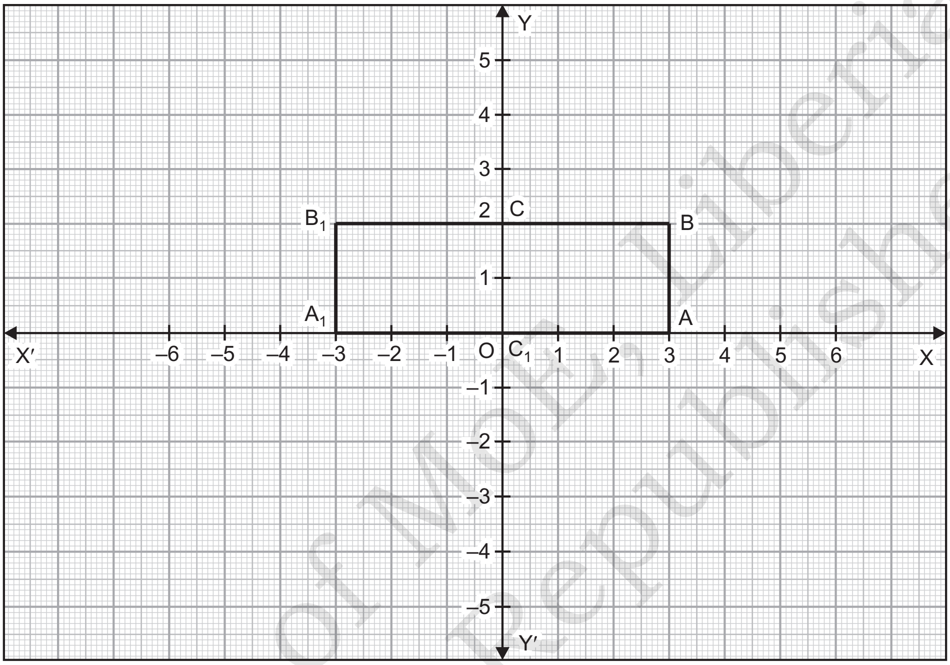
Reflection conserves angles, lengths and area but reserves the figure. To define a reflection you need to know the position of the line which the figure is to be reflected.

When a point is reflected in a line, the *image* point is at the opposite side of the line and the perpendicular distance from the point to the line is equal to the perpendicular distance from the image point to the line. The line called the *mirror line* or *line of reflection*.

i.e. object distance from mirror line = image distance from mirror line.

In figure OABC has been reflected in the y-axis to give O, A, B, C.
We shall consider reflections in the following mirror lines.

1. The x-axis or the line $y = 0$.
2. The y-axis or the line $x = 0$.



1. Reflection in the x-axis (i.e. reflection in the line $y = 0$):

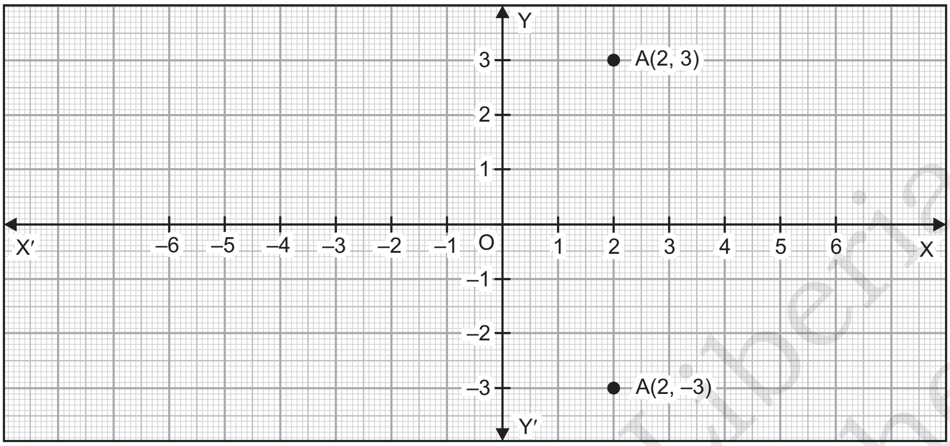
For reflection in the x-axis (horizontal axis), the x-axis serves as the mirror line.

Example 1. *Reflect the point X(2, 3) in the x-axis.*

Solution. Steps

1. With your x and y axes drawn. Locate the point A(2, 3) on the graph.
2. Measure the perpendicular distance from the mirror line (i.e., x-axis) to the point.
3. Measure the same distance as at the opposite side of the x-axis and locate your image point.
4. Write the coordinates to get your image A(2, –3) i.e., the x-coordinate is 2 and the y-coordinate is – 3. See Fig. for the graph.

Note: Write the x-coordinate first, followed by the y-coordinate.



Example 2. *A is the image of A(–2, –5) under a reflection in the x-axis. Find the coordinates of A.*

Solution. Under reflection in the x-axis, $(x, y) \rightarrow (x, -y)$

i.e. A(–2, –5) → A(–2, 5)

The coordinates of A is (–2, 5)

2. Reflection in the *y*-axis (*i.e.*, reflection in the line *x* = 0):

For reflection in the y-axis, the y-axis (vertical axis) serves as the mirror line.

Example 3. *Reflect the point C(2, 3) in the y-axis.*

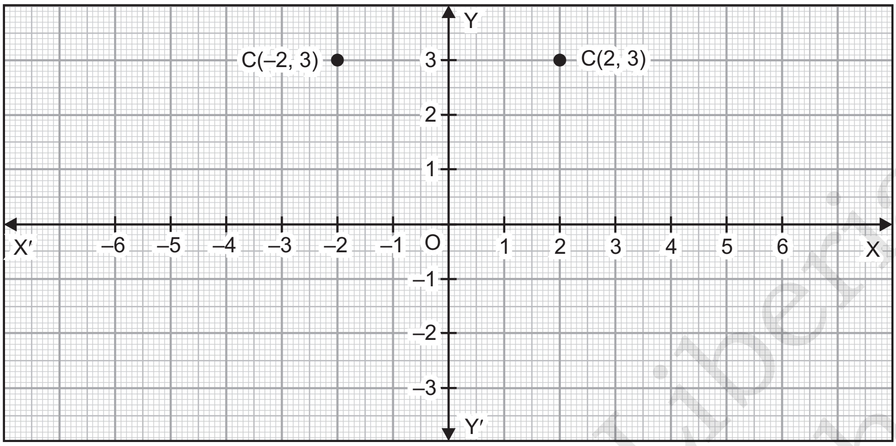
Solution. First method

For example when C(2, 3) is reflected in the y-axis we have C(– 2, 3) as the image point.

The steps are the same as reflected in the x-axis but here the y-axis serves as the mirror line.

Steps

1. With your axis drawn, locate the point C(2, 3) on the graph.
2. Measure the perpendicular distance from the mirror line (i.e., y-axis) to the point.
3. Measure the same distance at the opposite side of the y-axis and locate the image.
4. Write the coordinates to get your image C(– 2, 3). See figure for for the graph.



Example 4. *Using a scale of 2 cm to 1 unit, draw x and y axes for the interval $0 \leq x \leq 8$ and $-6 \leq y \leq 6$.*

(*i*) *Plot the points A(3, 1), B(1, 1) and C(1, 5) and described ABC.*

(*ii*) *Draw the $\Delta A_1B_1C_1$ which is the image of ΔABC under the reflection in the x-axis where $A \to A_1$, $B \to B_1$, $C \to C_1$. Indicate clearly the coordinates of $\Delta A_1B_1C_1$.*

Solution. Using the scale, draw the *x* and *y* axes in the intervals given on a graph sheet.

(*i*) Plot the points A, B and C on the graph and indicate clearly their coordinates. Join these points with straight lines to get ΔABC as shown in figure. ABC is a right-angled triangle.

(*ii*) The mapping for reflection in the *x*-axis is:

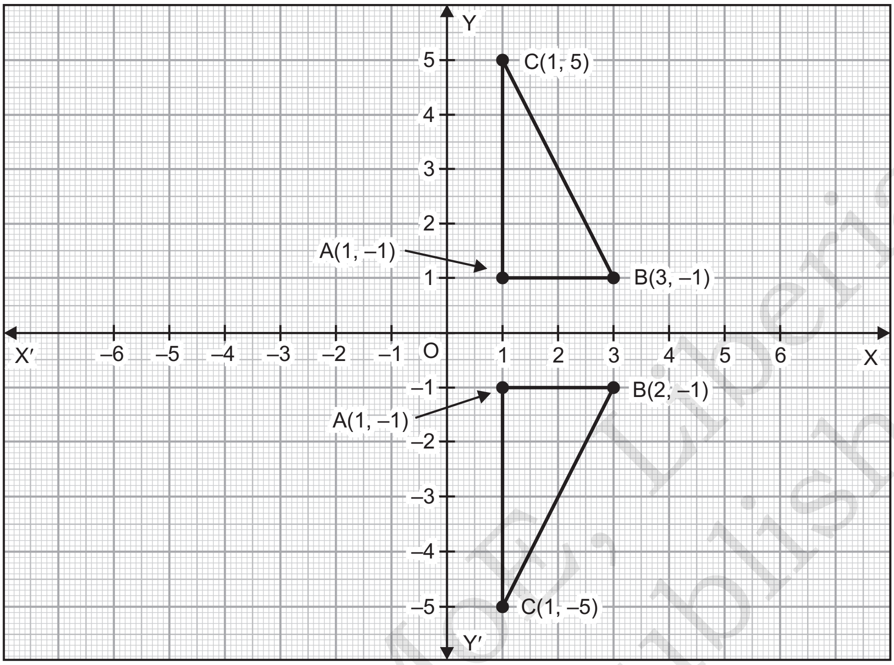
$$\begin{pmatrix} x \\ y \end{pmatrix} \text{ a } \begin{pmatrix} x \\ -y \end{pmatrix} \text{ or } (x, y) \to (x, -y)$$

$A(3, 1) \to A_1(3, -1)$

$B(1, 1) \to B_1(1, -1)$

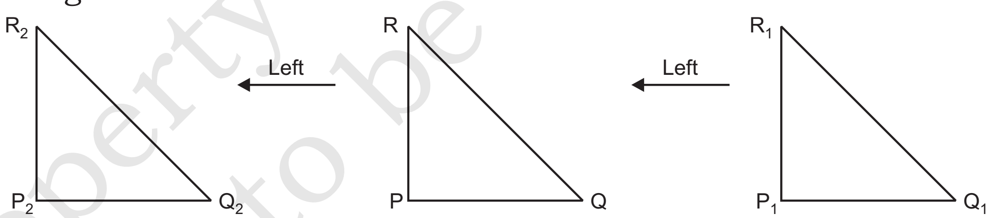
$C(1, 5) \to C_1(1, -5)$

Plot the points A_1, B_1 and C_1 on the graph to get $\Delta A_1B_1C_1$ and indicate their coordinates clearly as shown in figure.



Translation

If a figure is moved in a straight line by certain amounts in a certain direction without turning, and it still looks the same, the figure is said to have undergoen translation. Angles, lengths and areas remain unchanged.



The acitivity above shows that the movement of triangle ABC is a translation. When you walk from one place to another, you are making a translation. When you push a table away from a place or when you lift a bucket of water, you are making a translation. Give four real life examples of translation.

Translation of points

Translation can be described by a vector.

To translate a point P(2, 1) by a translation vector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, first plot the point on a graph sheet as shown in figure. Next move P 1 unit horizontally and 3 units vertically to the position P_1. The coordinates of this new position is (3, 4).

We can also use the relation below to find the coordinates of the new position. If the point (x, y) is translated by the vector $\begin{pmatrix} a \\ b \end{pmatrix}$.

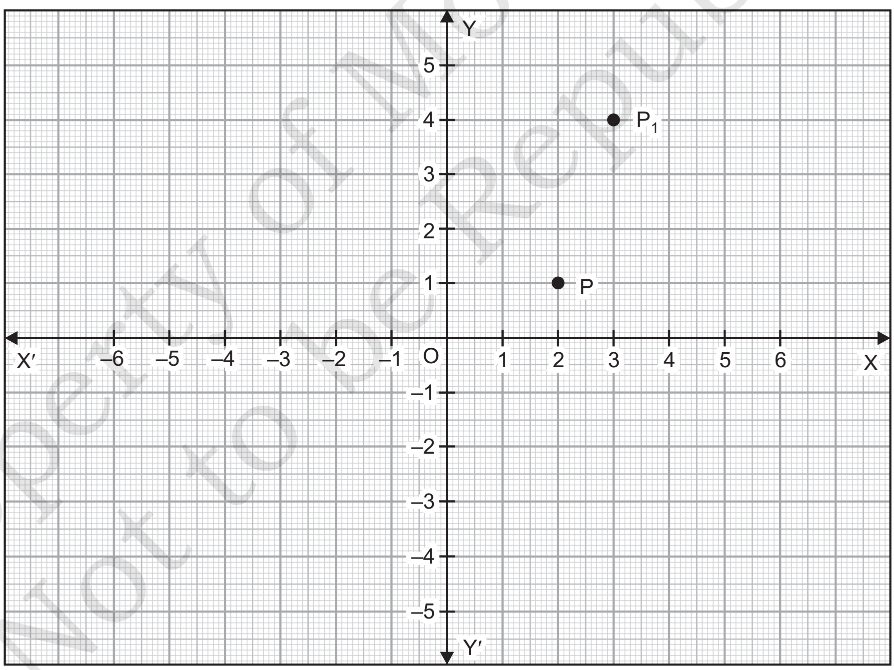
The mapping is:

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x + a \\ y + b \end{pmatrix}$$

or $$(x, y) \rightarrow (x + a, y + b)$$

The vector $\begin{pmatrix} a \\ b \end{pmatrix}$ vector will be given.

The image point is therefore found by adding the vector to the position vector of the point.



1. Find the image when the point and the translation vector are given.

Example 5. *Find the image A′ if A(3, 4) is translated by the vector* $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$.

Solution. Under a translation by the vector $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$,

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} d \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 + (-2) \\ 4 + 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix} \text{ or } (1, 9)$$

$\therefore$ The image of (3, 4) under the translation is (1, 9).

2. Finding the point when its image and the translation vector are given.

Example 6. *P(4, 6) is the image of a point P under the translation by the vector* $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. *Find the point P.*

Solution. If P has coordinates (*x*, *y*) then under a translation by the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, then

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} x + 1 \\ y + 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

Therefore from equality of vectors

$x + 1 = 4$ and $y + 2 = 6$ □ $x = 3$ and $y = 4$

$\therefore$ The point P has coordinates (3, 4).

Translation of plane figures

The translation of plane figures can be done in the Cartesian plane by translating the vertices of the plane figure by the given translation vector.

Example 7. *(a) Using a scale of 2 cm to 1 unit on each axies draw on a graph sheet two perpendicular axes OX and OY.*

(*b*) *On this graph, mark the x-axis from – 5 to 5 and the y-axis from – 5 to 5.*

(*c*) *Plot the point A(– 1, – 1), B(3, 4) and c(2, 1). Join the points to form a triangle.*

(*d*) *Draw the image of the triangle ABC under the translation by the vector* $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$ *and* $A \rightarrow A_1$, $B \rightarrow B_1$ *and* $C \rightarrow C_1$.

Solution. (*a, b*) Draw to perpendicular axes todivide the graph sheet into two equal parts and label each axis as shown in figure.

(*c*) Locate the points A(–1, –1), B(3, 4) and C(2, –1) and join them to get triangle ABC.

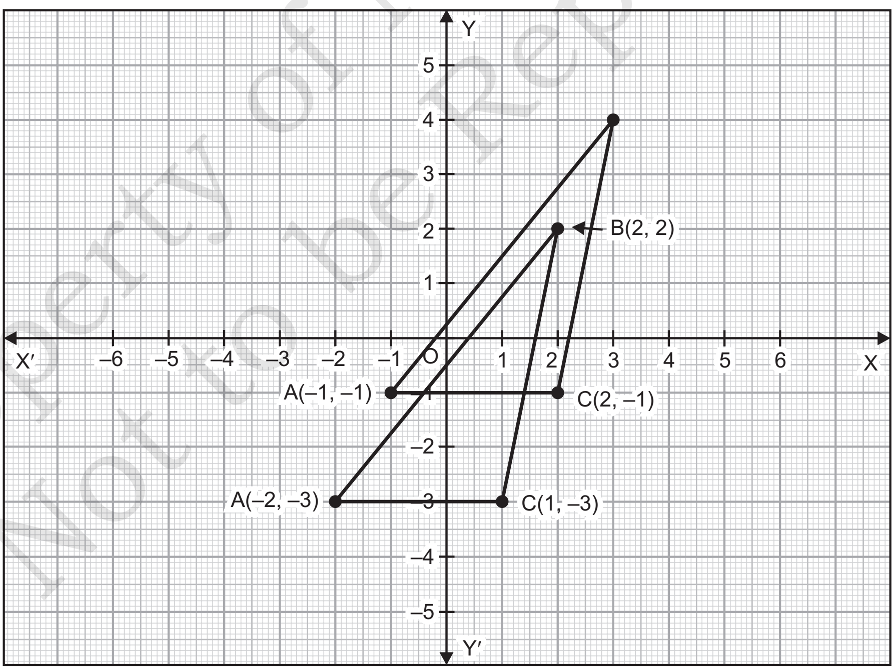
(*d*) Translate the points A, B and C by vector $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$ to get the image points A_1, B_1 and C_1.

$$A\begin{pmatrix} -1 \\ -1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \quad \text{i.e.} \quad A_1(-1, -3)$$

$$B\begin{pmatrix} 3 \\ 4 \end{pmatrix} \rightarrow \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{i.e.} \quad B_1(2, 2)$$

$$C\begin{pmatrix} 2 \\ -1 \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \quad \text{i.e.} \quad C_1(1, -3)$$

Locate the points A_1(–1, –3), B_1(2, 2) and C_1(1, –3) on the graph and join them to get triangle $A_1B_1C_1$.



Enlargement from the Origin O

If the point (*x, y*) is enlarged from the origin by a scale factor *k*, then image point is (*kx, by*)

The mapping is: $\begin{pmatrix} x \\ y \end{pmatrix} \to k\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ ky \end{pmatrix}$ or $(x, y) \to (kx, ky)$

k may be a positive or negative whole number or fraction.

Example 8. *Find the image of (3, 4) under the enlargement from the origin with scale factor –2.*

Solution. $\begin{pmatrix} 3 \\ 4 \end{pmatrix} \to 2\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -6 \\ -8 \end{pmatrix}$ or (–6, –8).

EXERCISE

1. A′ is the image of A(– 2, 4) under a reflection in the y-axis. Find the coordinates of A′.

2. Find the image of (*i*) (1, 2) and (*ii*) (–2, –4) under the translation by the vector $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$.

3. P(–2, 4) is the image of a point P(1, 2) under a translation by a vector. Find the translation vector.

4. The point (3, 4) is translated by the mapping $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 3x \\ y \end{pmatrix}$. Find its image.

5. Find the images of the following points under the mapping $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x + y \\ 3x + 2y \end{pmatrix}$

(*i*) (1, 2) (*ii*) (–2, 3)

6. Find the image of (3, –7) under the trnasformation $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2x + y \\ y - 3x \end{pmatrix}$.

TOPIC

18

Plane Geometry

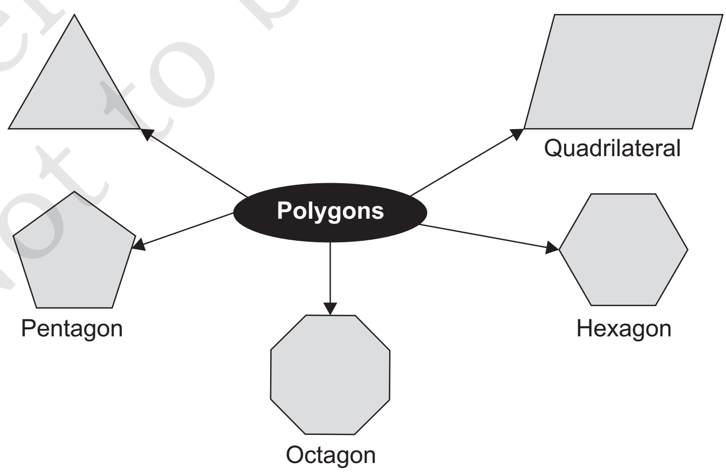
18.1. WHAT ARE POLYGONS?

A polygon is a closed figure made up of line segments (not curves) in a two-dimensionsal plane. Polygon is the combination of two words *i.e.,* poly (means many) and gon (means sides)

A minimum of three line segments is required to connect end to end, to make a closed figure. Thus a polygon with a minimum of three sides is known as Triangle and it is also called 3-gon. An *n*-sided polygon is called *n*-gon.

18.2. POLYGON SHAPE

By definition we know that the polygon is made up of line segments. Below are the shapes of some polygons that are enclosed by the different number of line segments.



18.3. TYPES OF POLYGON

Depending on the sides and angles, the polygons are classified into different types, namely

- Regular Polygon
- Irregular Polygon
- Convex Polygon
- Concave Polygon

Regular Polygon

If all the sides and interior angles of the polygon are equal, then it is known as a regular polygon.

The examples of regular polygons are square, rhombus, equilateral triangle etc.

Irregular Polygon

If all the sides and the interior angles of the polygon are of different measure, then it is known as an irregular polygon.

For example, a scalene triangle, a rectangle, a kite, etc.

Convex Polygon

If all the interior angles of a polygon are strictly less than 180 degrees, then it is known as an Convex polygon. The vertex will point outwards from the centre of the shape.

Concave Polygon

If one or more interior angles of a polygon are more than 180 degrees, then it is known as a concave polygon. A concave polygon can have atleast four sides. The vertex points towards the inside of the polygon.

However, a number of polygons are defined based on the number of sides, angles and properties.

18.4. ANGLES OF POLYGON

As we know any polygon has many vertices as it has sides. Each corner has a certain measures of angles. These angles are categorized into two types namely interior angles and exterior angles of a polygon.

18.5. INTERIOR ANGLE PROPERTY

The sum of all the interior angles of a simple n.gon = $(n - 2) \times 180$.

or

Sum = $(n - 2)n$ radius

where 'n' is equal to the number of sides of a polygon

For example, a quadrilateral has four sides, therefore, the sum of the all the interior angles is given by

Sum of interior angles of 4-sided polygon

$$= (4 - 2) \times 180° = 2 \times 180° = 360°$$

18.6. EXTERIOR ANGLE PROPERTY

The sum of interior and the corresponding exterior angles at each vertex of any polygon are supplementary to each other. For a polygon.

- Interior angle + Exterior angle = 180 degrees
- Exterior angle = 180 degrees – Interior angle

Properties

The properties of polygons are based on their sides and angles:

- The sum of all the interior angles of an n-sided polygon is $(n - 2) \times 180°$
- The number of diagonals in a polygon with n sides = $n(n - 3)/2$
- The number of triangles formed by joining the diagonals from one corner of a polygon = $n - 2$
- The measure of each interior angle of n-sided regular polygon is $[(n - 2) \times 180°]/n$
- The measure of each exterior angle of an n-sided regular polygon = $360°/n$

18.7. AREA AND PERIMETER FORMULAS

The area and perimeter of different polygons are based on the sides.

Area: Area is defined as the region covered by a polygon in a two-dimensional plane.

Perimeter: Perimeter of a polygon is the total distance covered by the sides of a polygon.

The formulas of area and perimeter for different polygons are given below:

Name of polygon	***Area***	***Perimeter***
Triangle	½ × (base) × (height)	$a + b + c$
Square	side 2	4(side)
Rectangle	Length × Breadth	2(Length + Breadth)
Parallelogram	Base × Height	2(Sum of pair of adjacent sides)
Trapezoid	Area = 1/2(sum of parallel side)height	Sum of all sides
Rhombus	½(Product of diagonals)	4 × sides
Pentagon	$\frac{1}{4}\sqrt{5(5+2\sqrt{5})}$ side^2	Sum of all five sides
Hexagon	3√3/2 (side)^2	Sum of all six sides

Let us see one of the frequently used and the primary types of polygon, *i.e.,* triangle.

18.8. TRIANGLES (3-GON)

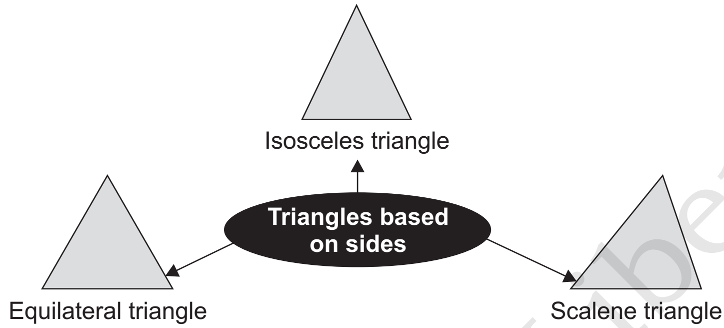
A triangle is the simplest form of the polygon that has three sides and three vertices. The triangles are also classified into different types, ababased on the sides and angles.

The sum of all the angles of the triangles is always equal to 180°(Straight angle).

18.9. TRIANGLES – BASED ON SIDES

- **Equilateral triangle:** Having all sides equal and angles of equal measure it is also called an equiangular triangle.
- **Isosceles triangle:** Having any 2 sides equal and angles opposite to the equal sides are equal
- **Scalene triangle:** Has all the 3 sides unequal.

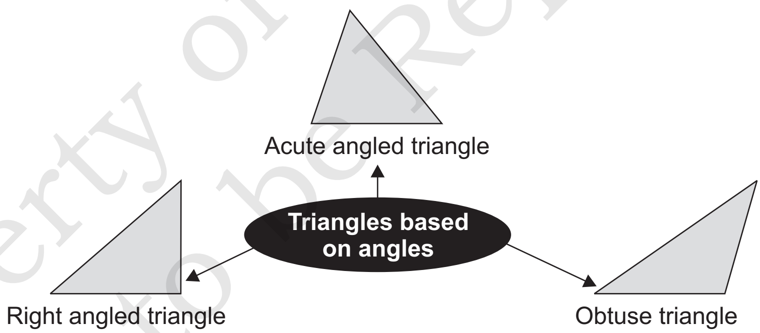
See the below figure to see the difference between the three types of triangles.



18.10. TRIANGLES – BASED ON ANGLES

- **Acute angled triangle –** Each angle is less than 90°.
- **Right angled triangle –** Any one of the three angles equal to 90°.
- **Obtuse angled triangle –** Any one angle is greater than 90°.

The below figure shows the three types of angles, based on angles.



Example 1. *A polygon is an octagon and its side length is 6 cm. Calculate its perimeter and value of one interior angle.*

Solution. The polygon is an octagon, so we have, $n = 8$

Length on one sides, $s = 6$ cm

The perimeter of the octagon

$$P = n \times s$$

$$P = 8 \times 6 = 48 \text{ cm}$$

Now, for the interior angle, we have

Interior angle of a regula polygon $= \dfrac{(n-2)180}{n}$

$= (8 - 2) \times 180/8$

$= 6 \times 180/8 = 135°$

Therefore, the perimeter of the octagon is 48 cm and the value of one of the interior angles is 135°.

Example 2. *Calculate the measure of one interior angle of a regular hexadecagon (16 sided polygon)?*

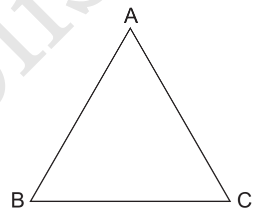
Solution. The polygon is an hexadecagon, so we have, $n = 16$

Interior angle of a regular polygon (IA) = $\frac{(n-2)180}{n}$

18.11. TRIANGLE

A plane figure bounded by three lines in a plane is called a triangle.

Let A, B, C be three points such that all are not in a line. then, the line segments AB, BC and CA form a triangle with vertices A, B and C. The segments AB, BC and CA are called the sides and the angles BAC, ABC and ACB are called the angles of ΔABC.



Types of Triangles

Triangles are classified into various types on the basis of the lengths their sides as well as on the basis of the measure of their angles.

Following are the types of triangles on the basis of sides.

Scalene Triangle: A triangle, no two of whose sides are equal is called a scalene triangle.

Isosceles Triangle: A triangle, two of whose sides are equal is length is called an isosceles triangle.

Equilateral Triangle: A triangle, all of whose sides are equal is called an equilateral triangle.

Following are the types of triangles on the basis of angles:

Acute Triangle: A triangle, each of whose sides is acute, is called an acute triangle or an acute angled triangle.

Right Triangle: A triangle with one angle a right angle is called a right triangle or a right angled triangle.

Obtuse Triangle: A triangle with one angle an obtuse angle, is known as an obtuse triangle or obtuse angled triangle.

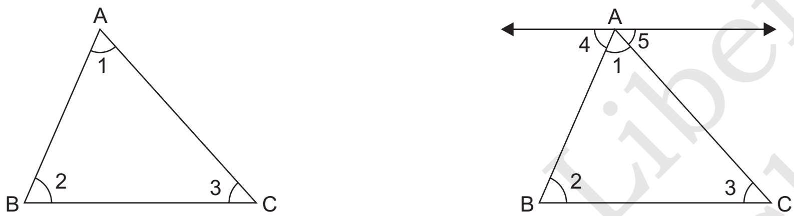
Note: It should be noted that an equilateral triangle is an isosceles triangle but the converse is not true.

18.12. ANGLE SUM PROPERTY OF A TRIANGLE

In this section, we shall deduce an important property about the tangles of a triangle, namely, that the sum of the tangles of a triangle is 180°.

Theorem 1. *The sum of three angles of a triangle is 180°.*

Given: A triangle ABC.



(i) (ii)

To Prove: $\angle A + \angle B + \angle C = 180°$ *i.e.* $\angle 1 + \angle 2 + \angle 3 = 180°$

Construction: Through A, draw a line l parallel to BC.

Proof: Since $l \parallel BC$. Therefore,

$$\angle 2 = \angle 4 \quad \text{[Alternate interior angles]}$$

and $$\angle 3 = \angle 5 \quad \text{[Alternate interior angles]}$$

$\therefore$ $$\angle 2 + \angle 3 = \angle 4 + \angle 5$$

$\Rightarrow$ $$\angle 1 + \angle 2 + \angle 3 = \angle 1 + \angle 4 + \angle 5$$

$\Rightarrow$ $$\angle 1 + \angle 2 + \angle 3 = \angle 4 + \angle 1 + \angle 5$$

$\Rightarrow$ $$\angle 1 + \angle 2 + \angle 3 = 180°$$

$$\left[\begin{array}{l} \because \ \text{Sum of angles at a point on a line is } 180° \\ \therefore \ \angle 4 + \angle 1 + \angle 5 = 180° \end{array}\right]$$

$\Rightarrow$ $$\angle A + \angle B + \angle C = 180°$$

Thus, the sum of the three angles of a triangle is 180°.

Corollary: *If the bisectors angles $\angle ABC + \angle ACB$ of a triangle ABC meet at a point O,*

then $$\angle BOC = 90° + \frac{1}{2}\angle A$$

Given: A ΔABC such that bisectors of $\angle ABC$ and $\angle ACB$ meet at a point O.

To Prove: $\angle BOC = 90° + \frac{1}{2}\angle A$

Proof: In BOC, we have

$$\angle 1 + \angle 2 + \angle BOC = 180°$$

In ΔABC, we have

$$\angle A + \angle B + \angle C = 180°$$

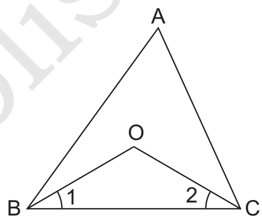
$$\Rightarrow \quad \angle A + 2(\angle 1) + 2(\angle 2) = 180°$$

$$\left[\begin{array}{ll} \because & BO \text{ and } CO \text{ are bisectors of } \angle ABC \text{ and } \angle ACB \text{ respectively} \\ \therefore & \angle B = 2\angle 1 \text{ and } \angle C = 2\angle 2 \end{array}\right]$$

$$\Rightarrow \quad \frac{\angle A}{2} + \angle 1 + \angle 2 = 90° \qquad \text{[Dividing both sides by 2]}$$

$$\Rightarrow \quad \angle 1 + \angle 2 = 90° - \frac{\angle A}{2} \qquad \ldots(ii)$$

Substituting this value of ∠1 + ∠2 in (*i*), we get



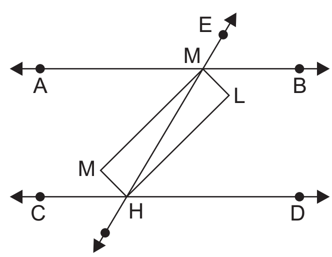
$$90° - \frac{\angle A}{2} + \angle BOC = 180°$$

$$\Rightarrow \quad \angle BOC = 180° - 90° + \frac{\angle A}{2}$$

$$\Rightarrow \quad \angle BOC = 90° + \frac{\angle A}{2}$$

Theorem 2. *If two parallel lines are interested by a transversal, prove that the bisector of the two pairs of interior angles enclose a rectangle.*

Given: Two parallel lines AB and CD and a transversl a EF intersecting them at G and H respectively. GM, HM, GL and HL are the bisectors of the two pairs of interior angles.



To Prove: GMHL is a rectangle.

Proof: We have

$$\angle AGH = \angle DHG \qquad \text{[Alternate interior angles]}$$

$$\Rightarrow \quad \frac{1}{2}\angle AGH = \frac{1}{2}\angle DHG$$

$$\Rightarrow \quad \angle HGM = \angle GHL$$

Thus, lines GM and HL are intersected by a transversal GHat G and H respectively such that pair of alternate angles are equal *i.e.*, ∠HGM = ∠GHL

$\because$ GM || HL

Similarly, we can prove that GL || HM. So, GMHL is a parallelogram.

Since AB || CD and EF is a transversal.

$$\therefore \quad \angle BGH + \angle DHG = 180°$$

$$\left[\because \text{Sum of interior angles on the same side of a transversal} = 180°\right]$$

$$\Rightarrow \quad \angle LGH + \angle LGH = 90°$$

$$\left[\because \frac{1}{2}\angle BGH = \angle LGH \text{ and, } \frac{1}{2}\angle DHG = \angle LHG\right]$$

But $\angle LGH + \angle LHG + \angle GLH = 180°$

[Sum of the angles of a triangle is 180°]

$$\therefore \quad 90° + \angle GLH = 180° \quad [\because \angle LGH + \angle LHG = 90°]$$

$$\Rightarrow \quad \angle GLH = 180° - 90°$$

$$\Rightarrow \quad \angle GLH = 90°$$

Thus, in the parallelogram GMHL, we have $\angle GLH = 90°$.

Hence, GMHL is a rectangle.

Example 3. *In ΔABC, $\angle B = 105°$, $\angle C = 50°$. Find $\angle A$.*

Solution. We have

$$\angle A + \angle B + \angle C = 180°$$

$$\Rightarrow \quad \angle A + 105° + 50° = 180°$$

$$\Rightarrow \quad \angle A = 180° - 155° = 25°$$

Example 4. *The sum of two angles of a triangle is equal to its third angle. Determine the measures of the third angle.*

Solution. Let ABC be a triangle such that

$$\angle A + \angle B = \angle C$$

We know that

$$\angle A + \angle B + \angle C = 180°$$

Putting $\angle A + \angle B = \angle C$ in (*ii*), we get

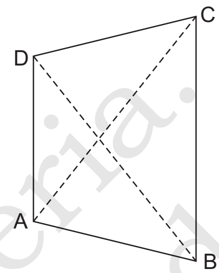
$$\angle C + \angle C = 180°$$

$$\Rightarrow \quad 2\angle C = 180°$$

$$\Rightarrow \quad \angle C = 90°$$

Thus, measure of the third angle is of 90°.

18.13. QUADRILATERAL



A four sided closed plane figure is called a quadrilateral.

Figure shows a quadrilateral ABCD in which AB, BC, CD and DA are the four sides. It is writeen as quad. ABCD or • ABCD.

The four points A, B, C and D are called its vertices.

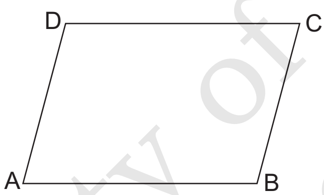
∠A, ∠B, ∠C and ∠D are the four angles of quad. ABCD.

AC and BD are called the diagonals of quad. ABCD.

18.14. VARIOUS TYPES OF QUADRILATERALS

(*i*) **Parallelogram:** A quadrilateral in which both pairs of opposite sides are parallel is called parallelogram. Figure shows a parallelogram.

(*ii*) **Rectangle:** A parallelogram in which each angle is 90°, is called a rectangle. Figure shows a rectangle.

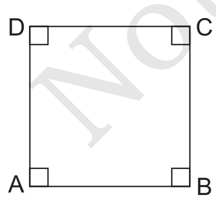


Parallelogram

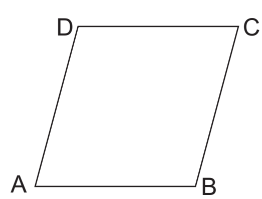
Rectangle

(*iii*) **Square:** A rectangle having all its sides equal is called a square. Figure shows a square in which AB = BC = CD = DA.

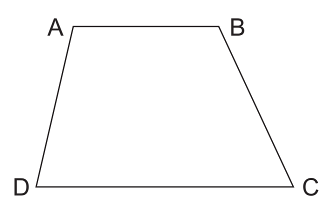
(*iv*) **Rhombus:** A parallelogram having all its sides equal is called a rhombus. Figure shows a rhombus in which AB = BC = CD = DA.



Square



Rhombus



Trapezium

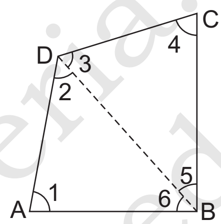
(*v*) **Trapezium:** A quadrilateral in which one pair of opposite sides is parallel is called a trapezium. Figure shows a trapezium is which AB || DC.

Theorem 3. *The sum of four angles of a quadrilateral is 360°.*

Given: ABCD is a quadrilateral

To Prove: ∠A + ∠B + ∠C + ∠D = 360°

Construction: Join BD



Proof: In ΔABD

∠1 + ∠2 + ∠6 = 180° (ASP) ...(*i*)

In ΔBCD,

∠3 + ∠4 + ∠5 = 180° (ASP) ...(*ii*)

Adding (*i*) and (*ii*), we get

∠1 + ∠2 + ∠6 + ∠3 + ∠4 + ∠5 = 180° + 180° = 360°

⇒ ∠1 + (∠2 + ∠3) + ∠4 + (∠4 + ∠6) = 360° (on rearranging)

⇒ ∠A + ∠D + ∠C + ∠B = 360°

∴ ∠A + ∠B + ∠C + ∠D = 360°

Example 5. *Three angles of a quadrilateral measure 56°, 115° and 84°. Find the measure of the fourth angle.*

Solution. Let the measure of the fourth angle be $x°$

We know sum of four angles of a quadrilateral = 360°

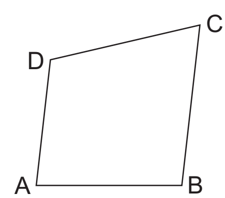
$56° + 115° + 84° + x° = 360°$

$255° + x° = 360°$

∴ $x° = 360° - 255° = 105°$.

Example 6. *The angles of a quadrilateral measure 2 : 4 : 5 : 7. Find the angles.*

Solution. In quadrilateral ABCD



⇒ ∠A = $2x$

⇒ ∠B = $2x$

⇒ ∠C = $5x$

⇒ ∠D = $7x$

We know

∠A + ∠B + ∠C + ∠D = 360°

$2x + 4x + 5x + 7x = 360°$

⇒ $18x = 360°$

∴ $x = 20°$

∠A = 2 × 20° = 40°; ∠B = 4 × 20° = 80°; ∠C = 5 × 20° = 100°;
∠D = 7 × 20° = 140°.

18.15. CIRCLE

A circle is a the locus of a point which moves in a plane in such a way that its distance from a given fixed point in the plane is always constant.

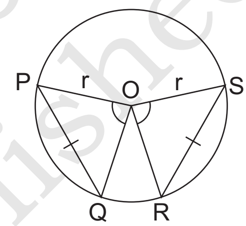
There are many circle theorem are given below:

Theorem 4. *Equal chords of a circle subtend equal angles at the centre.*

Given: A circle C(*o, r*) in which PQ and RS are two equal chords *i.e.*, PQ = RS

To Prove: ∠POQ = ∠ROS

Proof: In ΔPOQ and ΔROS,



OP = OS (Radii of a circle)

OQ = OR (Radii of a circle)

PQ = RS (given)

ΔPOQ ≅ ΔROS (By SSS)

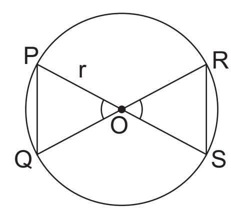
So ∠POQ = ∠ROS (By CPCT)

Theorem 5. *If the angles subtended by the chords of a circle at the centre are equal, then the chords are equal*

Given: A circle C(*o, r*) in which ∠POQ = ∠ROS.

To Prove: PQ = RS

Proof: In ΔPOQ and ΔROS,



OP = OS (Radii of a circle)

OQ = OS (Radii of a circle)

∠POQ = ∠ROS (given)

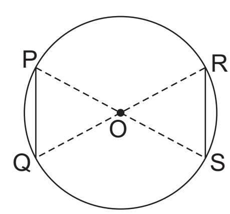
∴ ΔPOQ ≅ ΔROS (By SAS)

and so PQ = RS (By CPCT)

Theorem 6. *If two arcs of a circle are congruent, then the corresponding chords are equal.*

Given: A circle C(*o, r*) in which

$\overset{\frown}{PQ} \cong \overset{\frown}{RS}$ *i.e.*, ∠POQ = ∠ROS



To Prove: PQ = RS

Construction: Join OP, OQ, OR and OS

Proof: Case I: When $\overset{\frown}{PQ}$ and $\overset{\frown}{RS}$ are minor arcs

In $\Delta POQ = \Delta ROS$

$OP = OR$ (Radii of a circle)

$OQ = OS$ (Radii of a circle)

$\angle POQ = \angle ROS$ $(\because \overset{\frown}{PQ} \cong \overset{\frown}{RS})$

$\therefore$ $\Delta POQ \cong \Delta ROS$ (By SAS)

and so $PQ = RS$ (By CPCT)

Case II: When $\overset{\frown}{PQ}$ and $\overset{\frown}{RS}$ are major arcs

$\overset{\frown}{QP}$ and $\overset{\frown}{SR}$ are minor arcs

$\overset{\frown}{PQ} \cong \overset{\frown}{RS}$ (given)

$\Rightarrow$ $\overset{\frown}{QP} \cong \overset{\frown}{SR}$

$\Rightarrow$ $OP = SR$

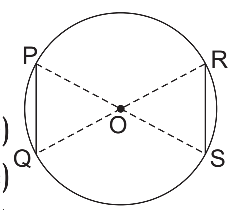
$\therefore$ $PQ = SR$

Theorem 7. *If two chords of a circle are equal then their corresponding arcs (minor, major or semicircular) are congruent.*

Given: A circle C(*o*, *r*) in which

To Prove: $\overset{\frown}{PQ} \cong \overset{\frown}{RS}$, where both $\overset{\frown}{PQ}$ and $\overset{\frown}{RS}$ are minor, major or semi-circular arc

Proof: Case I: When $\overset{\frown}{PQ}$ and $\overset{\frown}{RS}$ are minor arcs



In ΔPOQ and ΔROS

$OP = OR$ (Radii of a circle)

$OQ = OS$ (Radii of a circle)

$PQ = RS$ (given)

$\therefore$ $\Delta POQ \cong \Delta ROS$ (By SAS)

and so $\angle POQ = \angle ROS$ (By CPCT)

$\therefore$ $\overset{\frown}{PQ} \cong \overset{\frown}{RS}$

Case II: When $\overset{\frown}{PQ}$ and $\overset{\frown}{RS}$ are major arcs

$\overset{\frown}{QP}$ and $\overset{\frown}{SR}$ are minor arcs

$\therefore$ $\overset{\frown}{PQ} \cong \overset{\frown}{RS}$

$\Rightarrow$ $\overset{\frown}{QP} = \overset{\frown}{SR}$

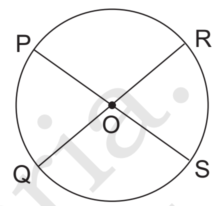
$$\Rightarrow \quad QP = SR$$

$$\Rightarrow \quad m(\widehat{QP}) = m(\widehat{SR})$$

$$\Rightarrow \quad 360° - m(\widehat{PQ}) = 360° - m(\widehat{RS})$$

$$\Rightarrow \quad m(\widehat{PQ}) = m(\widehat{RS})$$

$$\therefore \quad \widehat{PQ} \cong \widehat{RS}$$

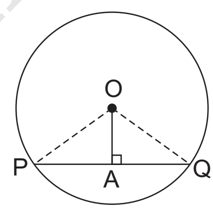


Case III: When PQ and RS are diameters. As PQ and RS are diameters, then $\widehat{PQ}$ and $\widehat{RS}$ are semicircles in which PQ = RS.

$\therefore$ They are congruent *i.e.*, $\widehat{PQ} \cong \widehat{RS}$

Theorem 8. *The perpendicular from the centre of a circle to a chord bisects the chord.*

Given: A circle C(*o*, *r*) in which PQ is a chord and QA ⊥ PQ



To Prove: PA = QA

Construction: Join OP and OQ

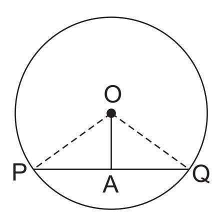
Proof: In ΔOAP and ΔOAQ

In ΔPOQ and ΔROS

	OA = OA	(Common)
	OP = OQ	(Radii of a circle)
	∠OAP = ∠OAQ	(Each 90°)
∴	ΔOAP ≅ ΔOAQ	(By RHS)
and so	PA = QA	(By CPCT)

Theorem 9. *The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.*

Given: A circle C(*o*, *r*) in which A is the midpoint of chord PQ i.e., PA = QA.



To Prove: OA ⊥ PQ

Construction: Join OP and OQ

Proof: In ΔOAP and ΔOAQ

	OP = OQ	(Radii of a circle)
	OA = OA	(Common)
	PA = QA	(Given)
∴	ΔOAP ≅ ∠OAQ	(By SSS)

and so $\angle OAP \cong \angle OAQ$ (By CPCT)

$\angle OAP + \angle OAQ = 180°$ (LPAs')

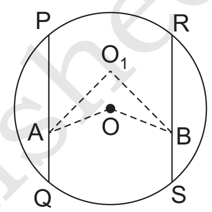
$\angle OAP + \angle OAP = 180°$

$2\angle OAP = 180°$

$\angle OAP = 90°$

Hence, $OA \perp PQ$

Corollary: *Prove that the perpendicular bisectors of two chord of a circle intersect at its centre.*



Given: A circle C(*o, r*) in which PQ and RS are two chords. Let O_1A and O_1B are perpendicular bisectors of PQ and RS respectively which intersect at O_1.

To Prove: O_1 coincides with O

Construction: Join OA and OB

Proof: A is the midpoint of PQ

($\because$ O_1A is the perpendicular bisector of PQ)

$\Rightarrow$ $OA \perp PQ$

$\Rightarrow$ OA is the perpendicular bisector of PQ

$\Rightarrow$ OA and O_1A both are perpendicular bisector of PQ

$\Rightarrow$ O_1A lies along OA

Similarly, B is the midpoint of RS.

$\Rightarrow$ $OB \perp RS$

$\Rightarrow$ OB is the perpendicular bisector of RS

$\Rightarrow$ OB and O_1B both are perpendicular bisector of RS

$\Rightarrow$ O_1B lies along OB

Thus O_1A lies along OA and O_1B lies along OB. This means that the point of intersection of O_1A and O_1B coincides with the point of intersection of OA and OB *i.e.,* O_1 coincides with O.

Hence the perpendicular bisector of PQ and RS intersect at the centre of the circle.

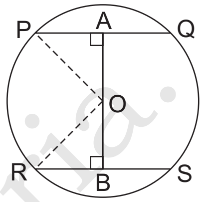
Theorem 10. *Equal chords of a circle are equidistant from the centre.*

Given: A circle C(*o, r*) in which PQ = RS

To Prove: OA = RS

Construction: Join OP and OR

Proof: We know that the perpendicular drawn from centre to a chord bisects the chord



$\therefore \qquad PA = \frac{1}{2}PQ$ and $RB = \frac{1}{2}RS$

$PQ = RS$ (Given)

$\Rightarrow \qquad \frac{1}{2}PQ = \frac{1}{2}RS$

$\therefore \qquad PA = RB$ (Radii of a circle)

$PA = RB$ (Proved)

$\angle OAP = \angle OBR$ (Each 90°)

$\therefore \qquad \angle OAP \cong \angle OBR$ (By RHS)

and so $\qquad OA = OB$ (By CPCT)

Theorem 11. *Chords equidistant from the centre of a circle are equal in length.*

Given: A circle C(*o*, *r*) in which OA ⊥ PQ and OB ⊥ RS. OA = OB.

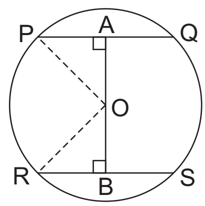
To Prove: PQ = RS

Construction: Join OP and OR

Proof: We know that the perpendicular drawn from centre to a chord bisects the chord

$\therefore \qquad PA = \frac{1}{2}PQ \Rightarrow PQ = 2PA$

$RB = \frac{1}{2}RS \Rightarrow RS = 2RB$



In ΔOAP and ΔOBR

$OP = OR$ (Radii of a circle)

$OA = OB$ (Given)

$\angle OAP = \angle OBR$ (Each 90°)

$PA = RB$

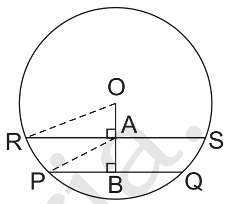
and so $\qquad 2PA = 2RB$

so $\qquad PQ = RS$

Example 7. *PQ and RS are two chords of a circle such that PQ = 6 cm and RS = 12 cm and PQ || RS. If the distance between OQ and RS and 3 cm, find the radius of the circle.*

Solution. PQ = 6 cm, RS = 12 cm

$$PQ \parallel RS, \; AB = 3 \text{ cm}$$



Let $OP = OR = r$

and $OA = x$

$$OB = OA + AB = x + 3$$

In ΔOAR $\quad OR^2 = OA^2 + RA^2$

$$r^2 = x^2 + 6^2 \qquad \left(\because RA = \frac{1}{2}RS\right)$$

$\therefore$ $\quad r^2 = x^2 + 36$

In ΔOBP $\quad OP^2 = OB^2 + PB^2$

$$r^2 = (x + 3)^2 + 3^2 \qquad \left(\because PB = \frac{1}{2}PQ\right)$$

$$x^2 + 36 = (x + 3)^2 + 9 \qquad \text{[Using } (i)\text{]}$$

$$x^2 + 36 = x^2 + 6x + 9 + 9$$

$\Rightarrow$ $\quad 36 = 6x + 18$

$\Rightarrow$ $\quad 36 - 18 = 6x$

$\Rightarrow$ $\quad 18 = 6x$

$\therefore$ $\quad x = \dfrac{18}{6} = 3 \text{ cm}$

Putting the value of x in (i), we get

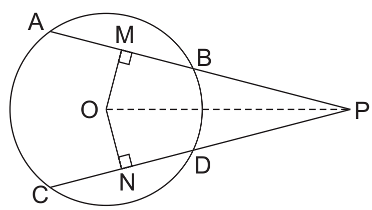
$$r^2 = 3^2 + 36$$

$$r^2 = 9 + 36 = 45$$

$$r = \sqrt{45} = 3\sqrt{5} \text{ cm}$$

Example 8. *Two equal chords AB and CD of a circle when produced intersect at a point P. Prove that PB = PD.*

Solution. Given: AB and CD are two equal chords of a circle



To Prove: PB = PD

Construction: Join OP

Proof: PM = ON

($\because$ AB = CD equal chords are equidistance from centre)

In ΔOMP and ΔONP

$$\angle M = \angle N \qquad \text{(Each } 90°\text{)}$$

$$OM = ON \qquad \text{(Proved)}$$

$$OP = OP \quad \text{(Common)}$$

$\therefore$ $$\Delta OMP \cong \Delta ONP \quad \text{(By RHS)}$$

and so $$PM = PN \quad \text{(By CPCT)}$$

$$AB = CD \quad \text{(Given)}$$

$$\frac{1}{2}AB = \frac{1}{2}CD$$

($\because$ Perpendicular drawn from centre to a chord bisects the chord)

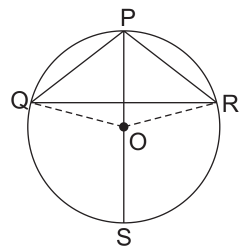
$$BM = DN \quad \ldots(ii)$$

Subtracting (*i*) and (*ii*), we get

$$PM - BM = PN - DN$$

$\therefore$ $$PB = PD$$

Example 9. *PQ and PR are equal chords drawn on opposite sides of a diameter PS. Prove that PS is the bisector of ∠QPR.*



Solution. Given: PQ and PR are two equal chords. PS is the diameter of the circle

To Prove: ∠OPQ = ∠OPR

Construction: Join QR, OQ and OR

Proof: In ΔPOQ and ΔPOR

$$PQ = PR \quad \text{(Given)}$$

$$OQ = OR \quad \text{(Radii of a circle)}$$

$$OP = OP \quad \text{(Common)}$$

$\therefore$ $$\Delta POQ \cong \Delta POR \quad \text{(By SSS)}$$

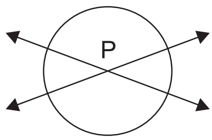
and so $$\angle OPQ = \angle OPR \quad \text{(By CPCT)}$$

18.16. NUMBER OF TANGENTS FROM A POINT ON A CIRCLE

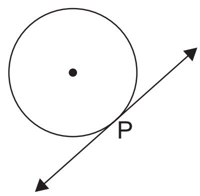
(*i*) If a point inside the circle it is not possible to draw any tangent to a circle through this point as shown in figure (*a*).

(*ii*) If a point is on the circle then only one tangent to the circle through this point can be drawn as shown in (*b*).

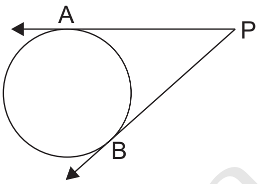
(*iii*) If a point is outside the circle then exactly two tangents can be drawn to the circle through this point as shown in (*c*).



(a)



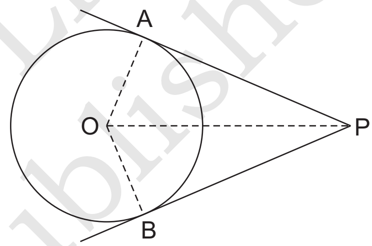
(b)



(c)

Theorem 12. *The lengths of tangents drawn from an external point to a circle are equal.*

Given: A circle with center O. PA and PB are tangents drawn from external point P to the circle.



To Prove: AP = BP

Construction: Join OA, OB andOP

Proof: We known that a tangent drawn from a point to a cricle is perpendicular to the radius at the point of contact.

$\therefore$ OA $\perp$ AP and OB $\perp$ BP.

In ΔOAP and ΔOBP

	$\angle A = \angle B$	(Each 90°)
	$OA = OB$	(Radius of the circle)
	$OP = OP$	(Common)
	$\Delta OAP \cong \Delta OBP$	(By RHS)
and so	$AP = BP$	(By CPCT)

Theorem 13. *If two tangents are drawn from an external point to a circle then:*

(*i*) *they subtend equal angles at the centre*

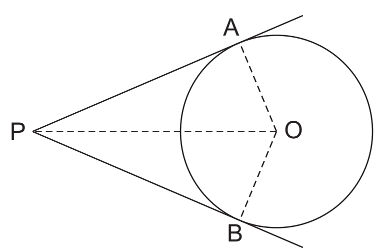
(*ii*) *they are equally inclined to the line joining the centre to the point.*

Given: A circle with center O. PA and PB are tangents drawn from point P to the circle.

To Prove: (*i*) $\angle AOP = \angle BOP$

(*ii*) $\angle APO = \angle BPO$

Construction: Join OA, OB andOP



Proof: We know that a tangent drawn from a point to a circle is perpendicular to the radius at the point of contact.

$\therefore$ OA $\perp$ AP and OB $\perp$ BP.

In ΔOAP and ΔOBP

$$OA = OB \quad \text{(Radius of a circle)}$$
$$AP = BP$$
$$(\because \text{ Tangents from an external point are equal})$$
$$OP = OP \quad \text{(Common)}$$
$$\therefore \quad \Delta OAP = \Delta OBP \quad \text{(By SSS)}$$
$$\text{and so} \quad \angle AOP = \angle BOP \quad \text{(By CPCT)}$$
$$\angle APO = \angle BPO \quad \text{(By CPCT)}$$

Example 10. *In a ΔABC, if $2\angle A = 3\angle B = 6\angle C$, determine $\angle A$, $\angle B$ and $\angle C$.*

Solution. We have,

$$2\angle A = 3\angle B = 6\angle C$$
$$\frac{\angle A}{3} = \frac{\angle B}{2} = \frac{\angle C}{1}$$

[Dividing throughout by 6 *i.e.* by the L.C.M of 2, 3 and 6]

$$\Rightarrow \quad \angle A : \angle B : \angle C = 3 : 2 : 1$$
$$\Rightarrow \quad \angle A = 3x, \angle B = 2x \text{ and } \angle C = x. \text{ Then,}$$
$$\Rightarrow \quad \angle A + \angle B + \angle C = 180°$$
$$\Rightarrow \quad 3x + 2x + x = 180°$$
$$\Rightarrow \quad 6x = 180°$$
$$\Rightarrow \quad x = 30°$$

Hence, measures of the angles of the triangle are

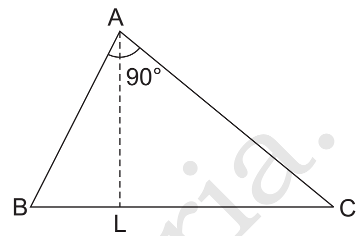
$$\Rightarrow \quad \angle A = 3x = 90°$$
$$\Rightarrow \quad \angle B = 2x = 60°$$
$$\Rightarrow \quad \angle C = x = 30°$$

Example 11. *A triangle ABC is right angled at A. AL is drawn perpendicular to BC. Prove that $\angle BAL = \angle ACB$.*

Solution. In AB ΔABL, we have

$$\angle BAL + \angle ALB + \angle B = 180°$$
$$\Rightarrow \quad \angle BAL + 90° + \angle B = 180° \quad [\because \ AL \perp BC \ \therefore \ \angle ALB = 90°]$$
$$\Rightarrow \quad \angle BAL + \angle B = 90°$$
$$\Rightarrow \quad \angle BAL = 90° - \angle B \quad \ldots(i)$$

In ΔABC, we have



$$\angle A + \angle B + \angle C = 180°$$

$$\Rightarrow \quad 90° + \angle B + \angle C = 180°$$

$$\Rightarrow \quad \angle B + \angle C = 180° - 90°$$

$$\Rightarrow \quad \angle B + \angle C = 90°$$

$$\Rightarrow \quad \angle C = 90° - \angle B$$

$$\Rightarrow \quad \angle ACB = 90° - \angle B \quad \ldots(ii)$$

From (*i*) and (*ii*), we get

$$\angle BAL = \angle ACB$$

EXERCISE

1. Calculate the measures of 1 exterior angle of a regular pentagon?

2. Find the sum of all the interior angle of a polygon having 29 sides.

3. Of the three angles of a triangle, one is twice the smallest and another is three times the smallest. Find the angles.

4. If the angles of a triangle are in the ratio 2 : 3 : 4, determine three angles.

5. The sum of two angles of a triangle is 80° and their difference is 20°. Find all the angles.

6. Find the length of tangent drawn to a circle with radius 5 cm from a point 13 cm away from the centre of the circle.

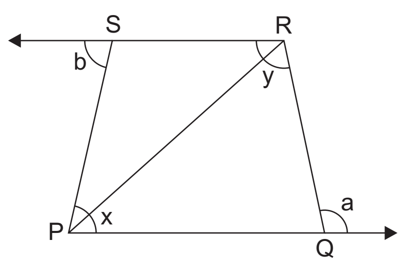
7. If the radii of the two concentric circles are 15 cm and 17 cm, show that the length of the chord of one circle which is tangent to other circle is 16 cm.

8. If the sum of the measure of the interior angle of polygon is 3240, find the number of sides of the polygon.

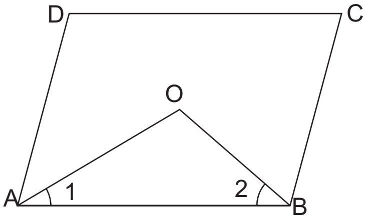
9. Find the sum of interior angles of a decagon.

10. Sum of all interior angles of a polygon is 3060°. How many sides does the polygon have?

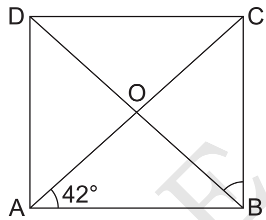
11. The sides PQ and RS of a quadrilateral PQRS are produced as shown in figure. Prove that $a + b = x + y$.



12. In a parallelogram, show that the angle bisectors of two adjacent angles intersect at right angles.



13. ABCD is a rectangle with ∠BAC = 42°. Determine ∠DBC.



PERIOD–V

M12CH19

TOPIC

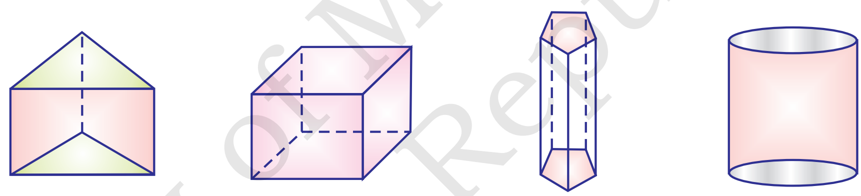
19

Solid Geometry

19.1. SURFACE AREA OF A PRISM

Prisms are solid with uniform cross-section. The cross-section may be triangular, rectangular, circular of shapes. A prism is named by the spade of its cross-section.

For example:



Triangular prism   Rectangular prism   Pentagonal prism   Cylinder

19.2. CUBOID

It is a prism with rectangular faces. All the sides of a cuboid are rectangles.

Formula used to calculate surface area of a cuboid is:

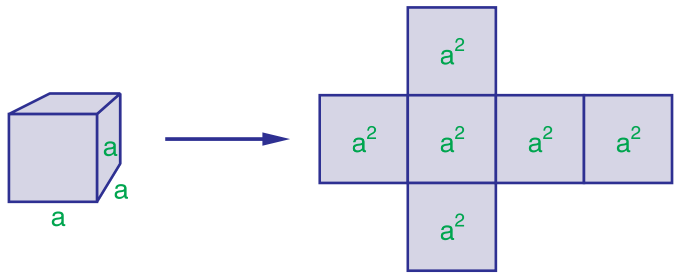
Surface area of a cuboid = $2(lb + bh + hl)$

where l, b, and h are respectively the three edges of the cuboid.

19.3. CUBE

Recall that a cuboid having equal length, equal length, breadth and height is called a cube. If each edge of the cube is 'a', the surface area of this cube would be

$2(a \times a + a \times a + a \times a)$ *i.e.,*   (see figure)



Formula used to calculate surface area of cuboid is

Surface area of a cube = $6a^2$

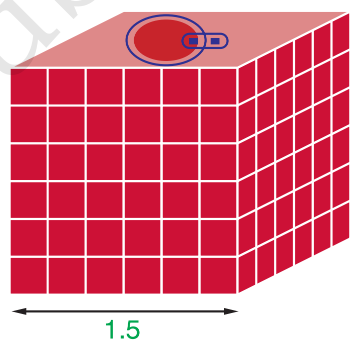
Example 1. *The length, breadth and height of a cuboid are 15 cm, 10 cm and 20 cm respectively. Find the surface area.*

Solution. Surface area of a cuboid = $2(lb + bh + hl)$

So, Surface area = $2[(15 \times 10) + (10 \times 20) + (20 \times 15)]$ cm^2

= $2(150 \times 200 + 300)$cm^2 = 2×650 cm^2 = **1,300 cm^2**

Example 2. *Runesha has built a cubical water tank with lid for his house, with each outer edge 1.5 m long. He gets the outer walls of the tank covered with square tiles of side 25 cm (see fig.) Find how much he will spend for the tiles, if the cost of the tiles is 3600* L$ *per dozen.*



Solution. Runesha is getting the four outer walls of the tank covered with tiles. He needs to calculate the surface area of the walls, to know the number of tiles required.

Edge of the cubical tank = 1.5 m = 150 cm

So, Surface area of the tank = $4 \times 150 \times 150$ cm^2

Area of each square tile = Side × Side = 25×25 cm^2

So, the number of tiles required

$$= \frac{\text{Surface area of the tank}}{\text{Area of each tile}} = \frac{4 \times 150 \times 150}{25 \times 25} = 144$$

Cost of 1 dozen tiles, i.e., 12 tiles = 3,600 L$

Therefore, the cost of one tile = $\frac{3,600}{12}$ L$ = 300 L$

So, the cost of 144 tiles = 144 × 300 L$ = 43,200 L$

19.4. VOLUME OF A CUBOID

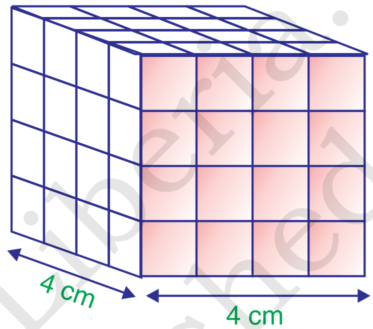
Solid objects occupy space. The measure of this occupied space is called the Volume of the object.

Formula used to calculate volume of a cuboid

= length × breadth × height

$\Rightarrow \quad V = l \times b \times h$

Now, consider figure it is a cube.



The cube is made of a number of unit cubes each of side 1 cm.

Let us count the number of unit cubes used in making this cube. It is found to be 64.

Volume of 1 unit cube = 1 cm^3

Volume of 64 unit cubes = 64 cm^3 = 4 cm × 4 cm × 4 cm = $(4\ cm)^3$

Formula used to calculate

Hence, volume of a cube = $(side)^3$

$\Rightarrow \quad V = a^3$

Example 3. *What is the volume of a rectangular solid with edge 9 m, 3.5 m and 2.8 m?*

Solution. Volume = $l \times b \times h$ = 9 m × 3.5 m × 2.8 m = **88.2 m^3**

Example 4. *A rectangular box has a volume of 9 m^3 and a base 50 cm by 25 cm. Find its height.*

Solution. Dimensions of the box are:

Volume (V) = 9 m^3; length (l) = 50 cm = 0.5 m; breadth (b) = 25 cm = 0.25 m

Since, volume = $l \times b \times h$

So, $$\text{height } (h) = \frac{\text{Volume}}{l \times b}$$

$$h = \frac{9\ m^3}{0.5\ m \times 0.25\ m} = 72\ m$$

$\therefore \quad h = 72$ m

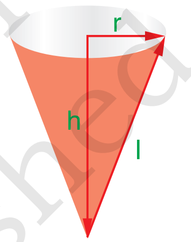
19.5. SURFACE AREA OF A RIGHT CIRCULAR CONE

Formula used to calculate curved surface area of a cone is

$$\text{Curved surface area of cone} = \frac{1}{2} \times l \times 2\pi r = \pi rl$$

where r is its base radius and l is its slant height.

Now, suppose h is the height of the cone. Then using Pythagoras theorem we have, $l^2 = r^2 + h^2$.



Therefore, $l = \sqrt{r^2 + h^2}$

If the base of the cone is closed, then a circular piece of paper of radius r is also required whose area is πr^2.

So, Formula used to calculate Total surface area of a cone is

$$\text{Total surface area of cone} = \pi rl + \pi r^2 = \pi r(l + r)$$

Example 5. *The slant height of a cone is 10 cm and the base radius is 7 cm. Find the curved surface area.*

Solution. Curved surface area = $\pi rl = \frac{22}{7} \times 7 \times 10 \text{ cm}^2 =$ **220 cm^2**

19.6. VOLUME OF A RIGHT CIRCULAR CONE

We observe that, three times the volume of the cone makes up the volume of a cylinder. Also, the cylinder has the same base radius and the same height as the cone has.

It means the volume of the cone = one-third the volume of the cylinder.

Formula used to calculate volume of a cone

So, $\quad \text{Volume of a cone} = \frac{1}{3}\pi r^2 h$

where r is the base radius and h is the height of the cone.

Example 6. *The height and the slant height of a cone are 21 cm and 28 cm respectively. Find the volume of the cone.*

Solution. By Pythagorean theorem $l^2 = r^2 + h^2$

$$\Rightarrow \qquad r = \sqrt{l^2 - h^2} = \sqrt{28^2 - 21^2} \text{ cm} = 7\sqrt{7} \text{ cm}$$

So, volume of the cone $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \times \frac{22}{7} \times 7\sqrt{7} \times 7\sqrt{7} \times 21 \text{ cm}^3$

$= \mathbf{7,546 \text{ cm}^3}$

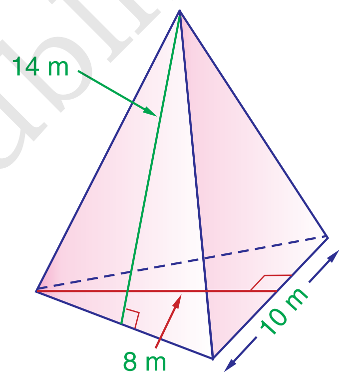
19.7. SURFACE AREA OF A PYRAMID

A pyramid has sides that are triangular-faced and a base. The base can be of any shape.

Formula used to calculate

$$S = l^2 + 2lh$$

Example 7. *Find the surface area of the regular pyramid shown in figure.*



Solution. To help visualize the surface area clearly, sketch a net. Then use the net to find the area of base and the area of each lateral face.

Area of base,

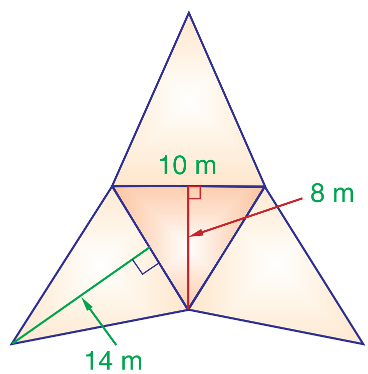
$A = \frac{1}{2} \times 10 \times 8$

$= 40 \text{ m}^2$

Area of each lateral face,

$A = \frac{1}{2} \times 10 \times 14$

$= 70 \text{ m}^2$



Finally, the surface area of the regular pyramid,

S = area of base

+ areas of lateral faces

$= 40 + \underbrace{70 + 70 + 70}$

$= \mathbf{250 \text{ m}^2}$

The pyramid has three congruent lateral faces. Count the area times.

The surface area is **250 square metre.**

19.8. VOLUME OF PYRAMID

Formula used to calculate volume of the pyramid

$$\text{Volume of the pyramid} = \frac{1}{3}\text{area of the base} \times \text{height}$$

19.9. SURFACE AREA OF A CYLINDER

This is the formula for the total surface area of a given cylinder whose radius is r and height is h

Total surface area of a cylinder (TSA) = $2\pi(h + r)$

19.10. VOLUME OF CYLINDER

Formula used to calculate

Volume of a cylinder = $\pi r^2 h$ cubic units

Example 8. *Find the total surface area of a container in a cylindrical shape whose diameter is 28 cm and the height is 15 cm.*

Solution. Given, diameter = 28 cm, so radius = 28/2 = 14 cm and height = 15 cm

By the formula of total surface are, we know

$$TSA = 2\pi r(h + r) = 2 \times (22/7) \times 14 \times (15 + 14)$$

$$TSA = 2 \times 22 \times 2 \times 29$$

$$TSA = 2552 \text{ sq. cm}$$

Hence, the total surface area of the container is 2552 sq. cm.

19.11. SURFACE AREA OF A SPHERE

A sphere is a three dimensional circular figure. All points on its surface are equidistant from its centre.

Formula used to calculate surface area of a sphere is

Surface area of a sphere = $4\pi r^2$

where r is the radius of the sphere.

Example 9. *Find the surface area of a sphere of radius 7 cm.*

Solution. The surface area of a sphere of radius 7 cm would be

$$4\pi r^2 = 4 \times \frac{22}{7} \times 7 \times 7 \text{ cm}^2 = \mathbf{616\ cm^2}$$

Formula used to calculate the volume of a sphere is

Volume of a sphere = $\frac{4}{3}\pi r^3$

where r is the radius of the sphere.

Example 10. *Find the volume of a sphere of radius 11.2 cm.*

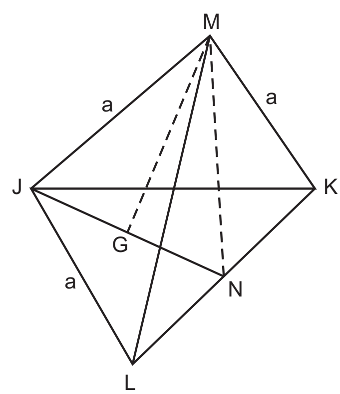
Solution. Required volume = $\frac{4}{3}\pi r^3 = \frac{4}{3} \times \frac{22}{7} \times 11.2 \times 11.2 \times 11.2 \text{ cm}^3$

= **5,887.32 cm³**

19.12. TETRAHEDRON

A pyramid on a triangular base is called a **tetrahedron**. In other words, a tetrahedron is a solid bounded by four triangular faces. Evidently a tetrahedron is a triangular pyramid. If the base of a tetrahedron is an equilateral triangle and the other triangular faces are isosceles triangles then it is called a **right tetrahedron**. A tetrahedron is said to be regular when all its four faces are equilateral triangles. Clearly, these equilateral triangles are congruent to one another.

A regular tetrahedron has been shown in the given figure. M is the vertex and the equilateral triangle JLK is the base of the regular tetrahedron. JL, LK, KJ, MJ, ML and MK are its six edges and three lateral faces are congruent equilateral triangles LKM, KJM and JLM. If G be the centroid of the base JLK and N, the mid-point of the side LK then MG is the height and MN, the slant height of the regular tetrahedron.



Let a be the length of an edge of a regular tetrahedron. Then,

1. Area of the slant height surface of the regular tetrahedron
   = sum of the areas of three congruent equilateral triangles
   = $3 \cdot (\sqrt{3})/4a^2$ square units.
2. Area of the whole surface of the regular tetrahedron
   = sum of the areas of four congruent equilateral triangles
   = $4 \cdot (\sqrt{3})/4a^2$ square units.
   = $\sqrt{3}a^2$ square units;

3. Volume of the regular tetrahedron

$= 1/3 \times$ area of the base $\times$ height

$= (1/3)\cdot(\sqrt{3})/4 \times a^2 \times (\sqrt{2})(\sqrt{3})a$ square units.

$= (\sqrt{2}/12)a^3$ cubic units.

Worked-out Problems in finding surface area and volume of a tetrahedron

Example 11. *Each edge of a regular tetrahedron is of length 6 metre. Find its total surface area and volume.*

Solution. A regular tetrahedron is bounded by four congruent equilateral triangles.

By question, each edge of the tetrahedron is of length 6 metre

Therefore, the total surface area of the tetrahedron

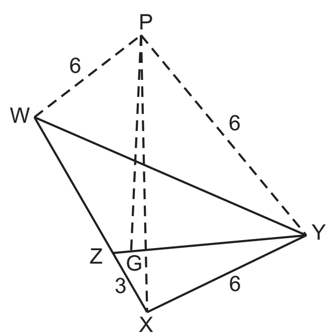
$= 4 \times$ area of the equilateral triangle of side 6 metres

$= 4 \times (\sqrt{3})/4 \cdot 6^2$ square metre

$= 36\sqrt{3}$ square metere

Therefore, from the right-angled ΔXYZ we get;

$YZ^2 = XY^2 - XZ^2 = 6^2 - 3^2$



[Since, $\overline{XY} = 6$ m (given) and

$\overline{XZ} = 1/2 \cdot \overline{WX} = 3$ m]

or $YZ^2 = 27$ or $\overline{YZ} = 3\sqrt{3}$

Let G be the centroid of the triangle WXY. Then,

$\overline{YG} = 2/3 \cdot \overline{YZ} = 2/3 \cdot 3\sqrt{3}$

Let $\overline{PG} \perp \overline{YG}$, hence from Δ PYG we get

$PG^2 = PY^2 - YG^2 = 6^2 - (2\sqrt{3})^2$ [Since $\overline{PY} = 6$m]

or $PG^2 = 36 - 12 = 24$ or $\overline{PG} = 2\sqrt{6}$

Therefore, the required volume of the tetrahedron

$= 1/3 \times$ (area of ΔWXY) $\times \overline{PG}$

$= 1/3 \cdot (\sqrt{3})/4 \cdot 6^2 \cdot 2\sqrt{6}$ cubic metre

$= 18\sqrt{2}$ cubic metre

19.13. WHAT IS THE SURFACE AREA OF A HEXAGONAL PRISM

The surface are of a hexagonal prism is defined as the total region covered by the surfaces of a hexagonal prism. Since, it has flat base, this it has a total surface area as well as a curved/lateral surface area. A hexagonal prism has 8 faces, 18 edges, and 12 vertices. It has equal top and bottom bases with diagonals crossing the center point of a regular hexagon.

The surface area of a hexagonal prism is expressed in square units. common units being square meters, square centimeters, square inches, etc. Just like other three-dimensional shapes, a hexagonal prism can also have two types of areas,

- Total Surface Area (TSA)
- Lateral Surface Area (LSA)

19.14. FORMULA OF SURFACE AREA OF HEXAGONAL PRISM

Total Surface Areas, TSA = 2(area of hexagon base) + 6(area of triangle face) sq. units = $6b(a + h)$ or $6ah + 3\sqrt{3}a^2$ (in case of regular hexagonal prism)

Lateral Surface Area, LSA = Ph = 6(area of the rectangle)

= $6ah$ sq. units.

where a = base length, a = apothem length, h = height

19.15. LATERAL SURFACE AREA OF A HEXAGONAL PRISM

The lateral surface area of the hexagonal prism is the sum of the area of 6 rectangular faces. Therefore, the lateral surface area, L = $6ah$ = $6ah$ aq.

19.16. TOTAL SURFACE AREA OF A HEXAGONAL PRISM

Total Surface Area, T = 6(area of rectangle face) + 2(area of hexagon base) sq. units = $6ah + 3\sqrt{3}a^2$

Example 12. *Determine the total surface area and the lateral surface area of a hexagonal prism with a base length of 4 inches and height of 11 inches.*

Solution. Given a = 4 inches and h = 11 inches

Lateral Surface Area of Hexagonal Prism = $6ah = 6 \times 4 \times 11 = 264$ square inches

Total Surface Area of Hexagonal Prism

$$= \text{LSA} + 3\sqrt{3}a^2 = 264 + 3\sqrt{3} \times (4)^2$$

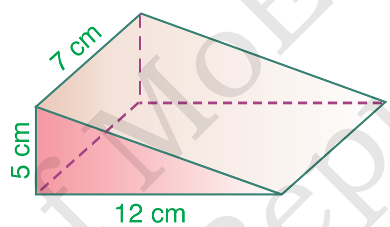
$\Rightarrow$ TSA = 264 + 83.136

$\Rightarrow$ TSA = 347.136 square inches

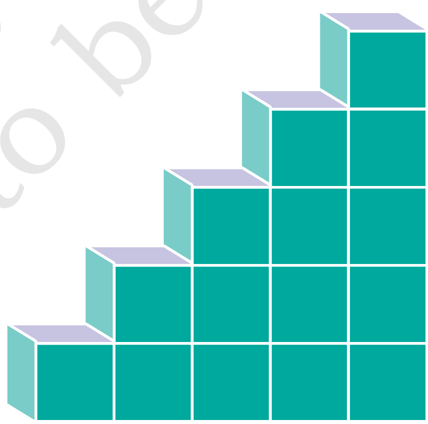
Therefore, the total surface area of the hexagonal prism is 347.136 square inches.

EXERCISE

1. What is the total surface area of this prism?

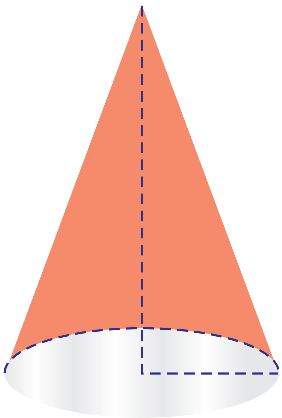


2. A child is playing with building blocks, which are of the shape of cubes. It has built a structure as shown in figure. The edge of each cube is 3 cm. Find the volume of the structure built by the child.

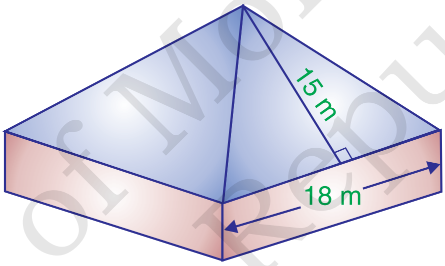


3. The height of a cone is 16 cm and its base radius is 12 cm. Find the curved surface area and the total surface area of the cone. (Use π = 3.14).

4. Find the volume of a cone (see figure) with a radius of 7 cm and a height of 30 cm.



5. The roof is shaped like a square pyramid. One bundle of shingles covers 20 square metre. How many bundles should you buy to cover the roof?



6. Find the total surface area of a hexagonal prism with the base of edge as 6 units and height as 12 units.

7. Find the height of the hexagonal prism if its total surface area is 396 sq feet, apothem length is 3 feet, base length is 6 feet.

PERIOD-VI

M12CH20

TOPIC

20 Probability and Statistics

20.1. FUNDAMENTAL PRINCIPLE OF COUNTING (FPC)

The fundamental principle of counting (FPC) states that if an operation can be performed in 'm' different ways, following which another operation can be performed in 'n' different ways, then both operations, in succession can be performed in exactly 'm × n' different ways.

Working Steps

Step I: *Identify the independent operations involved in the given problem.*

Step II: *Find the number of ways of performing each operation.*

Step III: *Multiply these numbers to get the total number of ways of performing all the operations.*

Example 1. *In a class there are 25 boys and 15 girls. The teacher wants to select 1 boy and 1 girl to represent the class in a function. In how many ways can the teacher make this selection?*

Solution. No. of ways of selecting one boy out of 25 boys = 25

No. of ways of selecting one girl out of 15 girls = 15

25 Boys
15 Girls

∴ By **FPC**, total number of ways of selecting one boy and one girl = 25 × 15 = **375**.

Example 2. *How many numbers are there between 100 and 1000 such that every digit is either 2 or 9?*

Solution A number between 100 and 1000 has 3 digits.

No. of ways of filling hundred's place = 2

(∵ Either 2 or 9 is to be used)

No. of ways of filling ten's place = 2

No. of ways of filling unit's place = 2

$\therefore$ By **FPC**, total number of numbers

$= 2 \times 2 \times 2 =$ **8**.

Place:	H	T	U
	↓	↓	↓
Ways:	2	2	2

20.2. FACTORIAL NOTATION

Let $n \in \mathbf{N}$. The continued product of first n natural numbers (beginning with 1 and ending with n) is called **factorial** n and is denoted by $n!$

Thus, $$\boldsymbol{n! = 1 \cdot 2 \cdot 3 \ \ldots\ldots\ . (n-1) \cdot n}$$

Illustrations $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

$8! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 = 40320.$

Factorial zero is defined as equal to 1 and we write $0! = 1$.

It is easily seen that $n! = n \cdot (n-1)!$

$= n(n-1) \cdot (n-2)!$

$= n(n-1)(n-2) \cdot (n-3)!$

....

$8! = 8 \times 7! = 8 \times 7 \times 6! = 56 \times 6!$ etc.

Example 3. *Evaluate:*

(*i*) $\dfrac{7!}{5!}$ (*ii*) $\dfrac{16 \cdot 15 \cdot 14 \cdot 13 \cdot 12!}{15!}$

(*iii*) $\dfrac{11!}{7!\,4!}$ (*iv*) $\dfrac{1}{5!} + \dfrac{1}{6!} + \dfrac{1}{7!}$.

Solution. (*i*) $\dfrac{7!}{5!} = \dfrac{7 \times 6 \times 5!}{5!} = \mathbf{42}$.

(*ii*) $\dfrac{16 \times 15 \times 14 \times 13 \times 12!}{15!} = \dfrac{16(15 \times 14 \times 13 \times 12!)}{15!} = \dfrac{16(15!)}{15!} = \mathbf{16}.$

(*iii*) $\dfrac{11!}{7!\,4!} = \dfrac{11 \times 10 \times 9 \times 8 \times 7!}{7!\,(4 \times 3 \times 2 \times 1)} = \dfrac{11 \times 10 \times 9 \times 8}{4 \times 3 \times 2} = \mathbf{330}$.

(*iv*) $\dfrac{1}{5!} + \dfrac{1}{6!} + \dfrac{1}{7!} = \dfrac{7 \times 6}{7 \times 6 \times 5!} + \dfrac{7}{7 \times 6!} + \dfrac{1}{7!} = \dfrac{42}{7!} + \dfrac{7}{7!} + \dfrac{1}{7!} = \mathbf{\dfrac{50}{7!}}$

Example 4. *If $x \in$ **N**, then solve the following equations:*

(*i*) $(x + 1)! = 12 \cdot (x - 1)!$ (*ii*) $\frac{1}{6!} + \frac{1}{7!} = \frac{x}{8!}$

Solution. (*i*) We have $(x + 1)! = 12 \cdot (x - 1)!$

$\Rightarrow \quad (x + 1)x \cdot (x - 1)! = 12 \cdot (x - 1)!$

$\Rightarrow \quad (x + 1)x = 12 \qquad [\because \ (x - 1)! \neq 0]$

$\Rightarrow \quad x^2 + x - 12 = 0$

$\Rightarrow \quad x = -4, 3 \quad \therefore \quad x = \mathbf{3}. \qquad (\because \ -4 \notin \mathbf{N})$

(*ii*) We have $\frac{1}{6!} + \frac{1}{7!} = \frac{x}{8!}$.

$$\Rightarrow \quad \frac{1}{6!} + \frac{1}{7 \cdot 6!} = \frac{x}{8 \cdot 7 \cdot 6!} \quad \Rightarrow \quad \frac{1}{6!}\left(1 + \frac{1}{7}\right) = \frac{1}{6!}\left(\frac{x}{56}\right)$$

$$\Rightarrow \quad \frac{8}{7} = \frac{x}{56} \quad \Rightarrow \quad x = \mathbf{64}.$$

20.3. PERMUTATIONS

An arrangement in a definite order of a number of things taking some or all at a time is called a **permutation.**

The total number of permutations of n distinct things taking $r \, (1 \leq r \leq n)$ at a time is denoted by ${}^nP_r$, where ${}^nP_r = \frac{n!}{(n-r)!}$, $0 \leq r \leq n$ or by $P(n, r)$. We define ${}^nP_0 = 1$.

where, ${}^nP_r = \frac{n!}{(n-r)!}$, $0 \leq r \leq n$.

Illustration. The permutations of 3 things *a, b, c* taking 2 at a time are:

ab *bc* *ca*

ba *cb* *ac.*

$\therefore \quad {}^3P_2 = 6$

The value of ${}^3P_2$ can also be found out by considering the problem of filling two places by using two out of *a, b, c.* Thus, the first place can be filled in 3 ways. After filling the first place, the second place can be filled in by using any of 2 (= 3 – 1) things.

$\therefore$ By **FPC**, the value of ${}^3P_2 = 3 \times 2 = 6$.

Working Rules

Rule I: ^nP_r (or $P(n, r)$) *denotes the number of permutations of n distinct things taking r at a time,* $1 \le r \le n$.

Rule II: *If value of r is given, then use:*

$$^nP_r = n(n-1)(n-2) \text{ .... } r \text{ factors.}$$

Rule III: *If value of r is not given, then use:* $^nP_r = \dfrac{n!}{(n-r)!}$.

Rule IV: $^nP_n = n! = n(n-1)(n-2) \text{ .... } 3 \cdot 2 \cdot 1$.

Example 5. *Evaluate:*

(*i*) $^5P_3$ (*ii*) $^7P_2$ (*iii*) $^{18}P_3$ (*iv*) $^6P_6$.

Solution. (*i*)

$$^5P_3 = \frac{5!}{(5-3)!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1}$$

$$= 5 \times 4 \times 3 = \mathbf{60}. \qquad \left(^nP_r = \frac{n!}{(n-r)!} \right)$$

(*ii*) $^7P_2 = 7 \times 6 = \mathbf{42}$.

(*iii*) $^{18}P_3 = 18 \times 17 \times 16 = \mathbf{4896}$.

(*iv*) $^6P_6 = \dfrac{6!}{(6-6)!} = \dfrac{6!}{0!} = \dfrac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{1} = \mathbf{720}$.

Example 6. *Show that:* $^{10}P_3 = {^9P_3} + 3\,{^9P_2}$.

Solution.

$$\text{LHS} = {^{10}P_3} = 10 \times 9 \times 8 = 720$$

$$\text{RHS} = {^9P_3} + 3\,{^9P_2} = (9 \times 8 \times 7) + 3(9 \times 8)$$

$$= 504 + 216 = 720$$

$\therefore$ LHS = RHS

Example 7. *Find n if:*

(*i*) $\dfrac{^nP_4}{^{n-1}P_4} = \dfrac{5}{3}$, $n > 4$ (*ii*) $^{n-1}P_3 : {^nP_4} = 1 : 9$.

Solution (*i*) We have $\dfrac{^nP_4}{^{n-1}P_4} = \dfrac{5}{3}$.

$$\Rightarrow \frac{n(n-1)(n-2)(n-3)}{(n-1)(n-1-1)(n-1-2)(n-1-3)} = \frac{5}{3} \Rightarrow \frac{n}{n-4} = \frac{5}{3}$$

$$\Rightarrow 3n = 5n - 20 \Rightarrow 2n = 20 \Rightarrow n = \mathbf{10}.$$

(*ii*) We have ${}^{n-1}P_3 : {}^{n}P_4 = 1 : 9$.

$$\Rightarrow \quad \frac{{}^{n-1}P_3}{{}^{n}P_4} = \frac{1}{9} \quad \Rightarrow \quad \frac{(n-1)(n-1-1)(n-1-2)}{n(n-1)(n-2)(n-3)} = \frac{1}{9}$$

$$\Rightarrow \quad \frac{1}{n} = \frac{1}{9} \quad \Rightarrow \quad n = \mathbf{9}.$$

20.4. COMBINATIONS

A selection (group) of a number of things taking some or all at a time is called a **combination**.

The total number of combinations of n distinct things taking $r(1 \le r \le n)$ at a time is denoted by ${}^{n}C_r$ or by $C(n, r)$. We define ${}^{n}C_0 = 1$.

where $${}^{n}C_r = \frac{n!}{r!(n-r)!}, \; 0 \le r \le n$$

Illustration. The combinations of 3 things a, b, c taking 2 at a time are:

$ab \qquad bc \qquad ca.$

$$\therefore \quad {}^{3}C_2 = 3$$

Working Rules

Rule I *${}^{n}C_r$ (or $C(n, r)$) denotes the number of combinations of n distinct things taking r at a time, $1 \le r \le n$.*

Rule II *If value of r is given, then use:*

$${}^{n}C_r = \frac{n(n-1)(n-2)......r \text{ factors}}{1 \cdot 2 \cdot 3 \, ...r}$$

Rule III *If value of r is not given, then use:* ${}^{n}C_r = \dfrac{n!}{r!(n-r)!}$.

Rule IV (*i*) ${}^{n}C_r = {}^{n}C_{n-r}$

(*ii*) *If ${}^{n}C_p = {}^{n}C_q$, then either $p = q$ or $p + q = n$.*

Rule V ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r, \; 1 \le r \le n.$

Rule VI ${}^{n}C_0 = 1$ *and* ${}^{n}C_n = 1.$

Example 8. *Evaluate the following:*

(*i*) ${}^{9}C_4$ (*ii*) ${}^{51}C_{49}$.

Solution. (*i*) $${}^{9}C_4 = \frac{9!}{4!(9-4)!} = \frac{9!}{4!5!} = \frac{9 \times 8 \times 7 \times 6 \times 5!}{4 \times 3 \times 2 \times 1 \times 5!} = \mathbf{126}.$$

Alternative method

$$^9C_4 = \frac{9 \times 8 \times 7 \times 6}{1 \times 2 \times 3 \times 4} = \mathbf{126}. \quad \left(\because \quad {}^nC_r = \frac{n(n-1)\ldots\ldots r \text{ factors}}{1 \times 2 \times \ldots\ldots \times r}\right)$$

(*ii*) $$^{51}C_{49} = {}^{51}C_{51-49} = {}^{51}C_2 = \frac{51 \times 50}{1 \times 2} = \mathbf{1275}. \quad (\because \quad {}^nC_r = {}^nC_{n-r})$$

20.5. REVIEW BASIC CONCEPT SETS

I. Finite and Infinite Sets

A set is said to be a **finite set** if it contains only finite number of elements. Otherwise, the set is said to be an **infinite set.**

If A is a finite set, then the number of elements in A is denoted by n(A) and called the **cardinality** of the finite set A.

Illustrations

1. The set {2, 4, 5, 10} is a finite set, because it contains only 4 elements.

2. The set of all even numbers is an infinite set.

II. Null Set

A set is said to be a **null set** if it does not contain any element. A null set is also called as *empty set* or *void set*. A null set is denoted by ϕ.

$$\therefore \qquad \phi = \{\}$$

Illustration 1. The set {0} is not a null set, because this set contains one element, namely '0'.

Illustration 2. Let A = $\{x : x \in \mathbf{N}, 2 < x < 3\}$. *A* does not contain any element, because there is no natural number between 2 and 3.

III. Singleton Set

A set is said to be a **singleton set** if it contains only one element.

Illustration 1 The sets {7}, {–15} are singleton sets.

Illustration 2 $\{x : x + 4 = 0, x \in \mathbf{Z}\}$ is a singleton set, because this set contains only one integer namely, –4.

Example 9. *Which of the following sets are singleton sets?*

(i) $\{x : x^2 + 2x + 1 = 0, x \in \mathbf{N}\}$ (ii) $\{x : x^2 = 9, |x| \leq 3, x \in \mathbf{N}\}$.

Solution (i) $x^2 + 2x + 1 = 0 \Rightarrow (x + 1)^2 = 0 \Rightarrow x + 1 = 0 \Rightarrow x = -1$

$x = -1$ is not a natural number.

$\therefore$ Given set = { }. This is not a singleton set.

(ii) $x^2 = 9$ implies $x = \pm 3$. $|x| \leq 3$ implies $x = -3, -2, -1, 0, 1, 2, 3$

$\therefore$ $x = -3, 3$ satisfy $x^2 = 9$ and $|x| \leq 3$ both.

Out of –3, 3, only $3 \in \mathbf{N}$.

$\therefore$ Given set = {3}. This is a singleton set.

IV. Equivalent Sets

Two sets A and B are said to be **equivalent sets** if the elements of A can be paired with the elements of B so that to each element of A there corresponds exactly one element of B and to each element of B there corresponds exactly one element of A.

Remark. Finite sets A and B are equivalent if and only if the number of elements in A and B are equal.

Illustration 1. The sets $\{a, b, c\}$ and {4, 7, 10} are equivalent sets.

Illustration 2. The sets {1, 2, 3, 4, 5, 6, ...} and {2, 4, 6, 8, 10, 12, ...} are equivalent sets because of the correspondence:

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \\ \downarrow & \downarrow & \downarrow & \downarrow & = = = = = = = \\ 2 & 4 & 6 & 8 & \end{array}$$

V. Equal Sets

Two sets are said to be **equal sets** if every element of one set is in the other set and *vice versa*.

In other words, the sets A and B are equal, if

$x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$.

If sets A and B are equal, then we write A = B.

Illustrations

1. Let A = {2, 3, 5, 6} and B = {2, 3, 4, 5, 6}. We have $A \neq B$, because $4 \in B$ and $4 \notin A$.

2. Let A = {2, 3, 4} and B = {2, 3, 4, 4, 4}. We have A = B.

Note. It is sufficient to write an element in a set only once.

3. Let A = {1, 4, 5} and B = {4, 1, 5}. We have A = B.

Note. Order of elements in a set is immaterial.

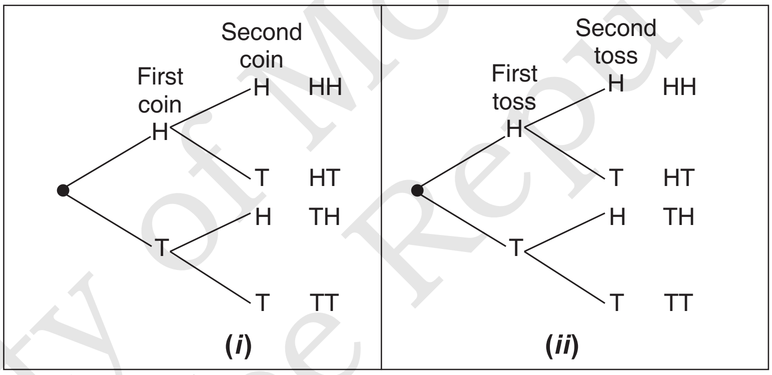
Remark. Equal sets are equivalent sets but equivalent sets may not be equal sets. For example, {2, 4, 5} and {3, 7, 8} are equivalent sets but not equal sets.

II. Tree Diagram

A **tree diagram** is a device used to enumerate all the logical possibilities of a sequence of events where each event can occur in a finite number of ways. A tree diagram is constructed from left to right and the number of branches at any point corresponds to the number of ways the next event can occur.

Illustration 1 The tree diagram of the sample space of the toss of two coins is shown in Fig. (*i*).

Illustration 2 The tree diagram of the sample space of the two tosses of a coin is shown in Fig. (*ii*).



Example 10. *A bag contains 4 red balls. What is the sample space if the random experiment consists of choosing:*

(*i*) *1 ball* (*ii*) *2 balls* (*iii*) *3 balls* (*iv*) *4 balls?*

Solution. Let the red balls be denoted by R_1, R_2, R_3 and R_4.

(*i*) In this experiment, one ball is drawn.

Number of elements in $S = {}^4C_1 = 4$

$\therefore$ $S = \{\mathbf{R_1}, \mathbf{R_2}, \mathbf{R_3}, \mathbf{R_4}\}$.

(*ii*) In this experiment, two balls are drawn.

Number of elements in $S = {}^4C_2 = \dfrac{4 \times 3}{1 \times 2} = 6$

$\therefore$ $S = \{\mathbf{R_1R_2}, \mathbf{R_2R_3}, \mathbf{R_3R_4}, \mathbf{R_4R_1}, \mathbf{R_1R_3}, \mathbf{R_2R_4}\}$.

(*iii*) In this experiment, three balls are drawn.

Number of elements in $S = {}^4C_3 = \dfrac{4\times3\times2}{1\times2\times3} = 4$

$\therefore$ $S = \{\mathbf{R_1R_2R_3, R_2R_3R_4, R_3R_4R_1, R_4R_1R_2}\}$.

(*iv*) In this experiment, 4 balls are drawn.

Number of elements in $S = {}^4C_4 = 1$

$\therefore$ $S = \{\mathbf{R_1R_2R_3R_4}\}$.

20.6. SET OPERATIONS

I. Venn Diagrams

The relationships between sets can be easily visualized by means of diagrams called **Venn diagrams.** Venn diagrams are named after the English logician *John Venn* (1834–1883).

II. Universal Set

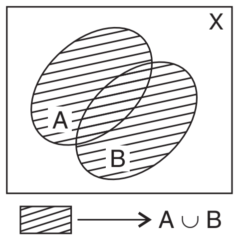
A set X is called a **universal set** if every set under consideration is a subset of X. The universal set is not a fixed set. It varies from situation to situation. A universal set is also denoted by 'U'.

In Venn diagrams, a universal set is depicted by the interior of a rectangle, whereas the subsets of universal set are depicted by the interior of circles, ellipses etc.

Illustration. The sets {1, 4}, {2, 3, 4}, {10, 19, 25} may be considered as subsets of universal set X = **N.** Here we may also take X = **Z**.

III. Union of Sets

The **union** of two sets A and B is defined as the set of all those elements which are in either A or B or both. The union of sets A and B is denoted as A ∪ B. The symbol '∪' is used to denote the 'union'.



In symbols, we write

$$\mathbf{A \cup B = \{x : x \in A \text{ or } x \in B\}}.$$

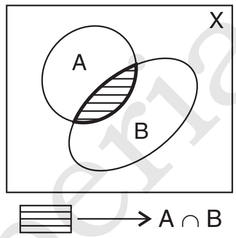
Illustrations

1. If A = {1, 2, 3, 4} and B = {2, 3, 4, 5, 6}, then

A ∪ B = {1, 2, 3, 4, 5, 6}.

IV. Intersection of Sets

The **intersection** of two sets A and B is defined as the set of all those elements which are in both A and B. The intersection of A and B is denoted by A ∩ B. The symbol '∩' is used to denote the 'intersection'.



In symbols, we write

$$\mathbf{A \cap B = \{x : x \in A \text{ and } x \in B\}}.$$

Illustrations

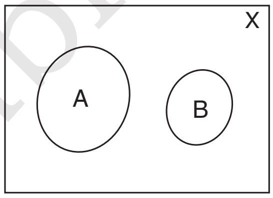
1. If A = {1, 2, 3, 4} and B = {2, 3, 4, 5, 6}, then A ∩ B = {2, 3, 4}.

2. If A = $\{x : x \in \mathbf{N},\ 0 < x < 5\}$ and B = $\{x : x \in \mathbf{N},\ 4 < x < 6\}$, then

$$A \cap B = \{1, 2, 3, 4\} \cap \{5\} = \phi.$$

V. Disjoint Sets

Two sets A and B are said to be **disjoint sets** if there is no element which belongs to both A and B. If A and B are disjoint sets, then A ∩ B = ϕ.

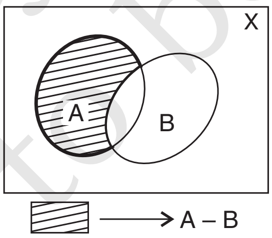


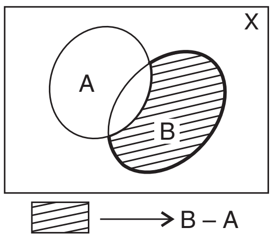
A ∩ B = ϕ

Illustration 1 The sets A = {4, 6, 10} and B = {7, 11, 15} are disjoint sets.

VI. Difference of Sets

The **difference** of two sets A and B in this order is the set of all those elements of A which are not in B. The difference of A and B in this order is denoted by A – B.





In symbols, we write $\mathbf{A - B = \{x : x \in A \text{ and } x \notin B\}}$

Similarly, B – A = $\{x : x \in B \text{ and } x \notin A\}$.

Remark. In general, A – B and B – A are not equal sets. These are always disjoint sets.

Example 11. *Find A – B and B – A if:*

(*i*) $A = \{2, 3, 6\}, B = \{1, 3, 7, 10\}$ (*ii*) $A = \{3, 6, 7\}, B = \{1, 2, 5, 8\}$.

Solution. (*i*) $A - B = \{2, 3, 6\} - \{1, 3, 7, 10\} = \mathbf{\{2, 6\}}$

and $B - A = \{1, 3, 7, 10\} - \{2, 3, 6\} = \mathbf{\{1, 7, 10\}}$

(*ii*) $A - B = \{3, 6, 7\} - \{1, 2, 5, 8\} = \mathbf{\{3, 6, 7\}}$

and $B - A = \{1, 2, 5, 8\} - \{3, 6, 7\} = \mathbf{\{1, 2, 5, 8\}}$.

Example 12. *If $A = \{4, 5, 8, 12\}$, $B = \{1, 4, 6, 9\}$ and $C = \{1, 2, 4, 7, 8, 10\}$, then find:*

(*i*) $A - (B - A)$ (*ii*) $A - (C - B)$.

Solution. (*i*) $B - A = \{1, 4, 6, 9\} - \{4, 5, 8, 12\} = \{1, 6, 9\}$

$\therefore$ $A - (B - A) = \{4, 5, 8, 12\} - \{1, 6, 9\} = \mathbf{\{4, 5, 8, 12\}}$

(*ii*) $C - B = \{1, 2, 4, 7, 8, 10\} - \{1, 4, 6, 9\} = \{2, 7, 8, 10\}$

$\therefore$ $A - (C - B) = \{4, 5, 8, 12\} - \{2, 7, 8, 10\} = \mathbf{\{4, 5, 12\}}$.

20.7. CONTIGENCY TABLE

Contigency tables (also called crosstable or two-way tables) are used in statistics to summarize the relationship between several categorical variables. A contigency table is a special type of frequency distribution table, where two variables as shown simultaneously.

Example:

	Pizza rolls	***Chips and Dip***	***Cookies***	***Totals***
Poker	10	3	12	25
Trivial pursuit	8	14	7	29
Monopoly 14	17	7	38	
WE Bowling	12	7	4	23
Totals	44	41	30	115

20.8. SAMPLE SPACE

The **sample space** of a random experiment is defined as the set of all possible outcomes of the experiment. The possible outcomes *i.e.,* the elements of the sample space are called **sample points.** The sample space is generally denoted by the letter *S*.

We list the sample space of some random experiments.

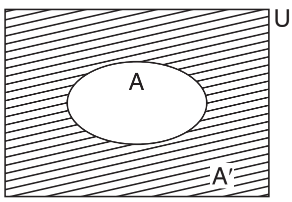
Random Experiment	**Sample Space**
1. Tossing of an unbiased coin	S = {H, T}
2. Tossing of an unbiased coin twice	S = {HH, HT, TH, TT} In *S*, the sample point *HT* represents 'head' on first toss and tail on second toss.
3. Tossing of two unbiased coins	S = {HH, HT, TH, TT} In S, the sample point HT represents 'head' on first coin and 'tail' on second coin.
4. Tossing of three unbiased coins	S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}*

Probability Event

An event is a collection of sample points with a common property. It is a subset of the sample space. For example, when a die is cast, the event E of throwing an even number is E = {2, 4, 6}. Also when two coins are tossed, the event of getting exactly one head E = {HT, TH}.

Complement of an Event

Let S be the sample space and A be an event of S. The se of all those sample points which are in S but not in A is called the complement of the event A and is denoted by $\bar{A}$. See figure. Note that the complement of a certain event is an impossible event.



Compound Events

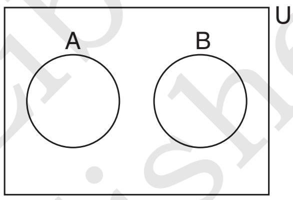
Events can be combined by the words 'or' and 'and'. Events which are thus combined are called compounds events. The words 'or' and 'and' correspond to '∪' and '∩' respectively in sets. Let A and B be two events. The compound events:

1. A ∪ B (*i.e.,* A or B) means eitherevent A or even B occur or both events occur.
2. A ∩ B (*i.e.,* A and B) means the events A and B occur together.

Note that "**or**" in probability means *addition*, "**and**" means multiplication

Mutually Exclusive Events

If A and B are events such that the two events cannot occur together or at the same time, then we say that the events A and B are **mutually exclusive** events. Thus if A ∩ B = ∅, then events A and B are mutually exclusive (see figure). For example, when you toss a coin, you can get either a head or a tail, not both at the same time. So the events of *getting a head* and *getting a tail* are mutually exclusive.



In this case, A and B are disjoint sets.

Independent Events

Two events A and B, are said to be independent if the occurrence of A does not affect the occurrence of B and vice versa. For example, if a fair die is thrown twice; let the event A = throwing a six on the first throw and B = throwing a siz on the second throw. The events A and B have no influence over each other and are therefore said to be independent.

What is a Conditional Event?

A conditional event algebra (CEA) contains not just ordinary events but alos conditional events, which have the form "if A, then B, The usual purpose of a CEA is to enable the defining of a probability function P, that satisfies the equation P(if A then B) = P(A and B)/P(A).

From now onward, we shall always assume that the outcomes of any given random experiment are *equally likely* unless the contrary is stated explicitely.

Example 13. *Three coins are tossed simultaneously. Write the sample space and the probabilities of getting (i) no head and (ii) two heads.*

Solution Here $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

(*i*) $$P(\text{no head}) = P(\{TTT\}) = \frac{n(\{TTT\})}{n(S)} = \frac{1}{8}.$$

(*ii*) $P(\text{two heads}) = P(\{HHT, HTH, THH\}) = \dfrac{3}{8}$.

20.9. 'ODDS INFAVOUR' AND 'ODDS AGAINST' AN EVENT

Let E be an event of a random experiment.

The ratio $P(E) : P(\bar{E})$ is called the **odds infavour** of happening of the event E.

The ratio $P(\bar{E}) : P(E)$ is called the **odds against** the happening of the event E.

Let odds in favour of an event E be $m : n$.

Let $P(E) = p$. $\therefore$ $P(\bar{E}) = 1 - p$

$\therefore$ $P(E) : P(\bar{E}) = m : n$ $\Rightarrow$ $p : 1 - p = m : n$

$\Rightarrow$ $\dfrac{p}{1-p} = \dfrac{m}{n}$ $\Rightarrow$ $np = m - mp$

$\Rightarrow$ $p = \dfrac{m}{m+n}$ *i.e.*, $P(E) = \dfrac{m}{m+n}$.

$\therefore$ **If odds infavour of E are $m : n$, then P(E) = $\dfrac{m}{m+n}$.**

Similarly, if odds against E are $m : n$, then odds infavour of E are $n : m$ and we have $P(E) = \dfrac{n}{n+m}$.

Example 14. *Find the probability of the event A if* (*i*) *odds infavour of event A are 5 : 7* (*ii*) *odds against A are 3 : 4.*

Solution. (*i*) Odds infavour of event A are 5 : 7.

Let $P(A) = p$. $\therefore$ $P(\bar{A}) = 1 - p$

$\therefore$ $p : 1 - p = 5 : 7$

$\Rightarrow$ $\dfrac{p}{1-p} = \dfrac{5}{7}$ $\Rightarrow$ $7p = 5 - 5p$

$\Rightarrow$ $12p = 5$ $\Rightarrow$ $p = \dfrac{5}{12}$

$\therefore$ $P(A) = \mathbf{\dfrac{5}{12}}$.

(*ii*) Odds against event A are 3 : 4. Let $P(A) = p$. $\therefore$ $P(\bar{A}) = 1 - p$

$\therefore$ $1 - p : p = 3 : 4$

$$\Rightarrow \quad \frac{1-p}{p} = \frac{3}{4} \quad \Rightarrow \quad 4 - 4p = 3p$$

$$\Rightarrow \quad 7p = 4 \quad \Rightarrow \quad p = \frac{4}{7}$$

$$\therefore \quad P(A) = \mathbf{\frac{4}{7}}.$$

Expected Value

Suppose the random variable x can take on the n values x_1, x_2, … x_n. Also suppose the probabilities that these values occur are respectively p_1, p_2, …, p_n. Then the expected value of the random variable is:

$$E(x) = x_1p_1 + x_2p_2 + \ldots + x_np_n$$

Measurse of Discrete Random Variables

Expected value of a discrete distribution

- An weighted average, taking into account the probability
- The expected value of random varibale x is denoted as $E(x)$

$$E(x) = \Sigma xiP(xi)$$

$$= x_1P(x_1) + x_2P(x_2) + \ldots + x_nP(x_n)$$

Example 15. *What is your expected gain when you play the flip-coin game twice?*

x	$P(x)$
–2	.25
0	.050
2	.25

$$E(x) = (-2) * 0.25 + 0 * 0.5 + 2 * 0.25$$
$$= 0$$

Your expected gain is 0! – a fair game.

EXERCISE

1. In a railway compartment, 6 seats are vacant on a beach. In how many ways can 3 passenegers sit on them?
2. Find the number of evetn positive numbers which have three digits.
3. Show that $n! + 1$ is not divisible by any number from 2 to n.

4. Prove that 33! is divisible by 2^{15}. What is the largest integer n such that 33! is divisible by 2^n?

5. Find n if:

(*i*) $\frac{{}^nP_4}{{}^{n-1}P_4} = \frac{5}{3}$, $n > 4$ (*ii*) ${}^{n-1}P_3 : {}^nP_4 = 1 : 9$.

6. Prove that: ${}^nP_r = {}^{n-1}P_r + r\,{}^{n-1}P_{r-1}$, $1 \le r \le n - 1$.

7. Find n, if ${}^{2n-1}P_n : {}^{2n+1}P_{n-1} = 22 : 7$.

8. Find n if:

(*i*) ${}^nC_8 = {}^nC_6$ (*ii*) ${}^nC_{n-4} = 5$

(*iii*) ${}^{25}C_{n+5} = {}^{25}C_{2n-1}$ (*iv*) ${}^{2n}C_3 : {}^nC_3 = 12 : 1$.

9. Find the value of:

(*i*) ${}^{15}C_3 + {}^{15}C_2$ (*ii*) ${}^{18}C_6 - {}^{17}C_6$

10. If ${}^nC_4$, ${}^nC_5$, ${}^nC_6$ are in A.P., find the value of n.

11. Which of the following sets are null sets?

(*i*) The set A of all prime numbers lying between 15 and 19

(*ii*) A = $\{x : x < 5, x > 6\}$ (*iii*) A = $\{x : x^2 = 16, x \in \mathbf{N}\}$.

12. Which of the following statements are true?

(*i*) If A = $\{x : x^2 = 4, x \in \mathbf{N}\}$, B = $\{-2\}$, then A ≠ B.

(*ii*) If A = $\{x : |x| < 2, x \in \mathbf{Z}\}$, B = $\{-1, 1\}$, then A = B.

(*iii*) If A = {1, 2, 3, 4, 5, 5}, B = {2, 1, 3, 4, 5}, then A = B.

(*iv*) If A = $\{x : x^2 - 5x + 7 = 0, x \in \mathbf{R}\}$, B = ϕ, then A = B.

13. The numbers 1, 2, 3 and 4 are written separately on four slips of paper. The slips are put in a box and mixed thoroughly. A person draws two slips from the box, one after the other, without replacement. Describe the sample space for the experiment.

14. A die is thrown repeatedly untill a six comes up. What is the sample space for this experiment?

15. Find A ∪ B if:

(*i*) A = {1, 2, 5}, B = {2, 3, 5, 7, 9} (*ii*) A = {3, 4, 7}, B = {1, 5, 6, 8}

(*iii*) A = ϕ, B = {2, 6, 8} (*iv*) A = {6, 7}, B = {1, 5, 6, 7, 9}.

16. Find A ∩ B if:

(*i*) A = {2, 3, 6}, B = {1, 3, 4, 6, 8}

(*ii*) A = {2, 3, 4, 8}, B = {1, 6}

(*iii*) A = ϕ, B = {3, 8, 11} (*iv*) A = {1, 3, 5}, B = {1, 2, 3, 4, 5, 6}.

PERIOD–VI

M12CH21

TOPIC

21

Exponential and Logarithmic Functions

21.1. EVALUATING EXPONENTIAL FUNCTIONS

Most of the functions (polynomial, rational, radical, etc.) we have studied thus far have been algebraic functions. Algebraic functions involve basic operations, powers, and roots. In this chapter, we discuss exponential functions and logarithmic functions. The following table illustrates the difference between algebraic functions and exponential functions:

Function	***Variable is in the***	***Constant is in the***	***Example***	***Example***
Algebraic	Base	Exponent	$f(x) - x^2$	$g(x) - x^{1/3}$
Exponential	Exponent	Base	$F(x) - 2^x$	$G(x) = \left(\frac{1}{3}\right)^x$

Definition: Exponential function

An exponential function with base b is denoted by

$$f(x) = b^x$$

where b and x are any real numbers such that $b > 0$ and $b \neq 1$.

Note:

- We eliminate $b = 1$ as a value for the base because it merely yields the constant function $f(x) = 1^x = 1$.
- We eliminate negative values for b because they would give non-real-number values such as $(-9)^{1/2} = \sqrt{-9} = 3i$.
- We eliminate $b = 0$ because 0^x corresponds to an undefined value when x is negative.

Sometimes the value of an exponential function for a specific argument can be found by inspection as an exact number.

x	-3	-1	0	1	3
$F(x) = 2^x$	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$2^0 = 1$	$2^1 = 2$	$2^3 = 1$

Example 1. *Let* $f(x) = 3^x$, $g(x) = \left(\frac{1}{4}\right)^x$ *and* $h(x) = 10x^2$. *Find the following values:*

(*a*) $f(2)$ (*b*) $f(\pi)$ (*c*) $g\left(-\frac{3}{2}\right)$

(*d*) $h(2.3)$ (*e*) $f(0)$ (*f*) $g(0)$

If an approximation is required, approximate to four decimal places.

Solution.

(*a*) $f(2) = 3^2 - \mathbf{9}$ (*b*) $f(\pi) = 3^\pi \approx \mathbf{31.5443}$*

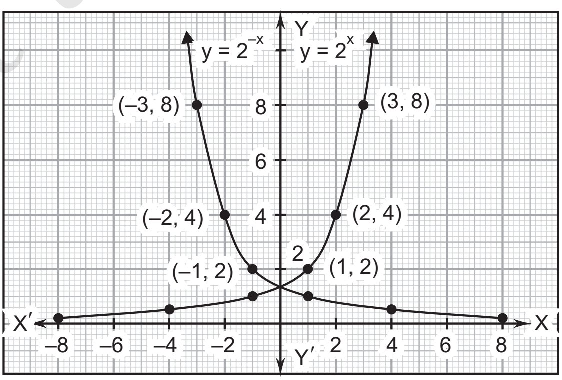
(*c*) $f\left(-\frac{3}{2}\right) = \left(\frac{1}{4}\right)^{-3/2} = 4^{3/2} = (\sqrt{4})^3 = 2^3 = \mathbf{8}$

(*d*) $h(2, 3) = 10^{2.3-2} = 10^{0.3} \approx \mathbf{1.9953}$

(*e*) $f(0) = 3^0 = \mathbf{1}$ (*f*) $g(0) = \left(\frac{1}{4}\right)^0 = \mathbf{1}$

21.2. GRAPHS OF EXPONENTIAL FUNCTIONS

Let's graph two exponential functions, $y = 2^x$ and, $y = 2^{-x} = \left(\frac{1}{2}\right)^2$ by plotting points.



x	$y = 2^x$	(x, y)
2	$2^{-3} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
– 1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	$2^0 = 0$	(0, 1)
1	$2^1 = 2$	(1, 2)
2	$2^2 = 4$	(2, 4)
3	$2^3 = 8$	(3, 8)

x	$y = 2^{-x}$	(x, y)
– 3	$2^{-(-3)} = 2^3 = 8$	(– 3, 8)
– 2	$2^{-(-2)} = 2^2 = 4$	(– 2, 4)
– 1	$2^{-(-1)} = 2^1 = 2$	(– 1, 2)
0	$2^0 = 1$	(0, 1)
1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(1, \frac{1}{2}\right)$
2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(2, \frac{1}{4}\right)$

Notice that both graphs' y-intercept is (0, 1) (as shown to the left) and neither graph has an x-intercept. The x-axis is a horizontal asymptote for both graphs. The following box summarizes general characteristics of the graphs of exponential functions.

21.3. PROCEDURE FOR GRAPHING $f(x) = b^x$

Step 1: Label the point (0, 1) corresponding to the y-intercept $f(0)$.

Step 2: Find and label two additional points corresponding to $f(-1)$ and $f(1)$.

Step 3: Connect the three points with a smooth curve with the x-axis as the horizontal asymptote.

Example 2. *Graphing Exponential Functions for $b > 1$.*

Graph the function $f(x) = 5^x$.

$f(0) = 5^x = 1$

$f(0) = 5^1 = 1$

$f(-1) = 5^{-1} = \frac{1}{5} = 0.2$

Solution. Step 1: Label the y-intercept (0, 1)

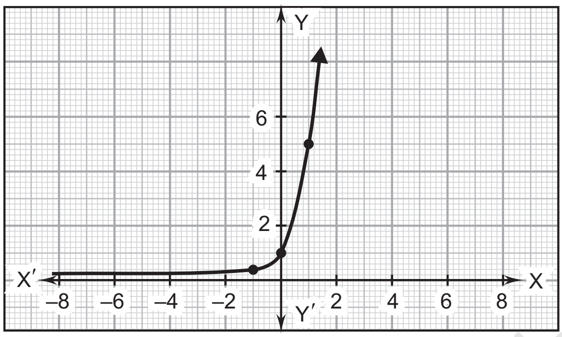
Step 2: Label the point (1, 5).

Label the point (–1, 0.2).

Step 3: Sketch a smooth curve through the three points with the x-axis as a horizontal asymptote.

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$



21.4. THE NATURAL BASE e

Any positive real number can serve as the base for an exponential function. A particular irrational number, denoted by the letter *e*, appears as the base in many applications, as you will soon see when we discuss continuous compounded interest. Although you will see 2 and 10 as common bases, the base that appears most often is e, because e, as you will come to see in your further studies of mathematics, is the natural base. The exponential function with base *e*, $f(x) = e^x$, is called the exponential function or the natural exponential function. Mathematicians did not pull this irrational number out of a hat. The number *e* has many remarkable properties, but most simply, it comes from evaluating the expression $\left(1+\frac{1}{m}\right)^m$ as *m* gets large (increases without bound).

	$\boldsymbol{m}$	$\left(\boldsymbol{1}+\frac{\boldsymbol{1}}{\boldsymbol{m}}\right)^{\boldsymbol{m}}$
	1	2
	10	2.59374
$e \approx 2.71828$	100	2.70481
	1000	2.71692
	10000	2.71815
	100000	2.71827
	1000000	2.71828

Calculators have an e^x button for approximating the natural exponential function.

Example 3. *Evaluating the Natural Exponential Function*

Evaluate $f(x) = e^x$ for the given x-values. Round your answers to four decimal places.

(*a*) $x = 1$ (*b*) $x = –$

(*c*) $x = 1.2$ (*d*) $x = –0.47$

Solution.

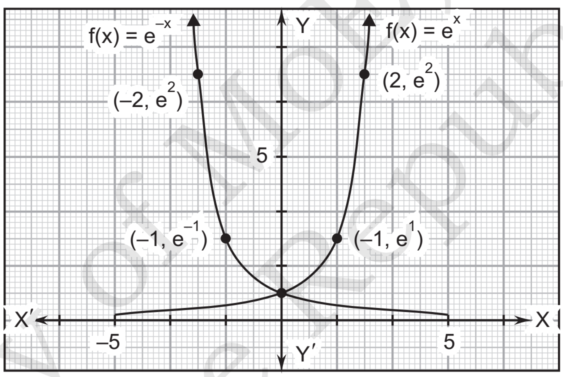
(*a*) $f(1) = e^1 \approx 2.718281828 \approx$ **2.7183**

(*b*) $f(– 1) = e^1 \approx 0.367879441 \approx$ **0.3679**

(*c*) $f(1.2) = e^{1.2} \approx 3.320116923 \approx$ **3.3201**

(*d*) $f(– 0.47) = e^{0.47} \approx 0.625002268 \approx$ **0.6250**

Like all exponential functions of the form $f(x) = b^x$, $f(x) = e^x$ and $f(x) = e^{-x}$ have (0, 1) as their y-intercept and the x-axis as a horizontal asymptote as shown in the figure on the right.



21.5. APPLICATIONS OF EXPONENTIAL FUNCTIONS

Exponential functions describe either growth or decay. Populations and investments are often modeled with exponential growth functions, while the declining value of a used car and the radioactive decay of isotopes are often modeled with exponential decay functions. In Section, various exponential models will be discussed. In this section, we discuss doubling time, half-life, and compound interest.

21.6. DOUBLING TIME GROWTH MODEL

The doubling time growth model is given by

$$P = P_0 2^{t/d}$$

where P = Population at time t, P_0 = Population at time $t = 0$

d = Doubling time

Note that when $t = d$, $P = 2P_0$ (population is equal to twice the original).

The units for P and P_0 are the same and can be any quantity (people, dollars, etc.). The units for t and d must be the same (years, weeks, days, hours, seconds, etc.).

In the investment scenario with Maria and David, P_0 = \$5,000 and d = 10 years, so the model used to predict how much money the original \$5,000 investment yielded is $P = 5000(2)^{t/10}$. Maria retired 40 years after the original investment, t = 40, and David retired 30 years after the original investment, t = 30.

Maria: $P = 5000(2)^{40/10} = 5000(2)^4 = 5000(16) = 80,000$

David: $P = 5000(2)^{30/10} = 5000(2)^3 = 5000(8) = 40,000$

Example 4. *In 2004 the population in Kazakhstan, a country in Asia, reached 15 million. It is estimated that the population doubles in 30 years. If the population continues to grow at the same rate, what will the population be in 2024? Round to the nearest million.*

Solution. Write the doubling model. $P = P_0 2^{t/d}$

Substitute P_0 = 15 million, d = 30 years, and t = 20 years.

$P = 15(2)^{20/30}$

Simplify, $P = 15(2)^{2/3} \approx 23.8110$

In 2024, there will be approximately 24 million people in Kazakhstan.

Example 5. *The radioactive isotope of potassium 42K, which is used in the diagnosis of brain tumors, has a half-life of 12.36 hours. If 500 milligrams of potassium 42 are taken, how many milligrams will remain after 24 hours? Round to the nearest milligram.*

Solution. Write the half-life formula. $A = A_0\left(\frac{1}{2}\right)^{t/h}$

Substitute A_0 = 500 mg, h = 12.36 hours, t = 24 hours.

$$A = 500\left(\frac{1}{2}\right)^{24/12.36}$$

Simplify $A \approx 500(0.2603) \approx 130.15$

After 24 hours, there are approximately 130 milligrams of potassium 42 left.

21.7. SIMPLE INTEREST

Simple interest was defined where the interest I is calculated based on the principal P, the annual interest rate r, and the time t in years, using the formula I = Prt.

If the interest earned in a period is then reinvested at the same rate, future interest is earned on both the principal and the reinvested interest during the next period. Interest paid on both the principal and interest is called compound interest.

21.8. COMPOUNT INTEREST

If a principal P is invested at an annual rate *r* compounded *n* times a year, then the amount A in the account at the end of t years is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

The annual interest rate *r* is expressed as a decimal.

The following list shows the typical number of times interest is compounded:

Annually	$n = 1$	Monthly	$n = 12$
Semiannually	$n = 2$	Weekly	$n = 52$
Quarterly	$n = 4$	Daily	$n = 365$

Example 6. *If $3,000 is deposited in an account paying 3% compounded quarterly, how much will you have in the account in 7 years?*

Solution. Write the compound interest formula.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Substitute P = 3000, $r = 0.03$, $n = 4$, and $t = 7$.

$$A = 3000\left(1 + \frac{0.03}{4}\right)^{(4)(7)}$$

Simplify

You will have $3,698.14 in the account.

21.9. THE NUMBER e

The number e is defined as the value that $(1 + 1/n)^n$ approaches as n becomes large. (In calculus this idea is made more precie through the concept of a limit). The table shows the values of the expression $(1 + 1/n)^n$ for increasingly large values of n.

m	$\left(1+\frac{1}{n}\right)^n$
1	2.00000
5	2.48832
10	2.59374
100	2.70481
1000	2.71692
10000	2.71815
100000	2.71827
1000000	2.71828

It appears that rounded to five decimal places, $e = 2.71828$; in fact, the approximate value to 20 decimal places is

$$e \approx 2.71828182845904523536$$

It can be shown that e is an irrational number, so we cannot write its exact value in decimal form.

21.10. THE NATURAL EXPONENTIAL FUNCTION

The number e is the base for the natural exponential function. Why use such a strange base for an exponential function? It might seem at first that a base such as 10 is easier to work with. We will see, however that in certain applications the number e is the best possible base. In this section we study how e occurs in the description of compound interest.

The natural exponential function is the exponential finction

$$f(x) = e^x$$

with base e. It is often referred to as the exponential function.

Since $2 < e < 3$, the graph of the natural exponential function lies between the graphs of $y = 2^x$ and $y = 3^x$ as shown in Figure.

Scientific calculations have a special key for the function $f(x) = e^x$. We use this key in the next example.

Example 7. *Evaluate each expression rounded to five decimal places*

(*a*) e^3 (*b*) $2e^{-0.53}$ (*c*) $e^{4.8}$

Solution. We use e^x key on a calculator to evaluate the exponential function.

(*a*) $e^3 \approx 20.08554$ (*b*) $2e^{-0.53} \approx 1.17721$

(*c*) $e^{4.8} \approx 121.51042$

21.11. LOGARITHMIC FUNCTIONS

Recall from section that a function *f* has an inverse function if and only if *f* is one-to-one. In the preceding section, you learned that $f(x) = a^x$ is one-to-base, so *f* must have an inverse function. This inverse function is the logarithmic function with base *a* and is denoted by the symbol $\log_a x$ read as "log base *a* of *x*".

Defintion of Logarithmic Function with Base a

The equations $y = \log_a x$ and $x = a^y$ are equivalent, where the first equation is written in logarithmic form and the second equation is written in exponential form. For example, $2 = \log_3 9$ is equivalent to $9 = 3^2$ and $5^3 = 12$ is equivalent to $\log_5 125 = 3$.

When evaluating logarithms, remember that a *logarithm is an exponent*. This means that $\log_a x$ is the exponent to which *a* must be raised to obtain *x*. For example, $\log_2 8 = 3$ because 2 raised to the third power is 8.

Example 8. *Evaluate each logarithm at the given value of x.*

(*a*) $f(x) = \log_2 x, x = 32$ (*b*) $f(x) = \log_3 x, x = 1$

(*c*) $f(x) = \log_4 x, x = 2$ (*d*) $f(x) = \log_{10} x, x = \frac{1}{100}$

Solution. (*a*) $f(32) = \log_2 32 = 5$ because $2^5 = 32$

(*b*) $f(1) = \log_3 1 = 0$ because $3^0 = 1$

(*c*) $f(2) = \log_4 2 = \frac{1}{2}$ because $4^{1/2} = \sqrt{4} = 22$

(*d*) $f\left(\frac{1}{100}\right) = \log_{10}\frac{1}{100} = -2$ because $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$

21.12. PROPERTIES OF LOGARITHMS

1. $\log_a 1 = 0$ because $a^0 = 1$
2. $\log_a a = 1$ because $a^1 = a$
3. $\log_a a^x = x$ and $a^{\log_a x} = x$ Inverse properties
4. If $\log_a x = \log_a y$, then $x = y$ One-to-One Property

Example 9. *Using properties of Logarithms*

(*a*) *To solve* $\log_3 x = \log_3 12$ *for x use the One-to-One property.*

(*b*) *To solve* $\log (2x + 1) = \log 3x$ *for x use the One-to-One property.*

Solution. (*a*)

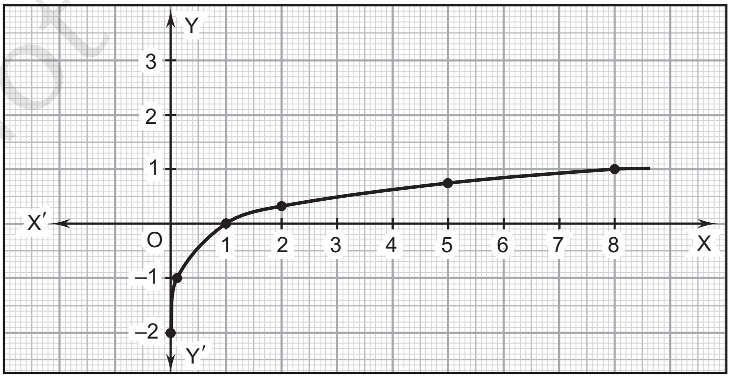
	$\log_3 x = 12$	Write original equation
	$x = 12$	One to One property
(*b*)	$\log (2x + 1) = \log 3x$	Write original equation
	$2x + 1 = 3x$	One to One property
	$1 = x$	Subtract $2x$ from each

Example 10. *Sketch the graph of f(x) = log x. Identify the vertical asymptote.*

Solution. Being by constructing a table of values. Note that some of the values can be obtained without calculator by using the properties of logarithms. Other required a calculator.

	Without calculator				***With calculator***		
x	$\frac{1}{100}$	$\frac{1}{10}$	1	10	2	5	8
$f(x) = \log x$	– 2	– 1	0	1	0.301	0.699	0.903

Next, plot the points and connect them with a smooth curve, as shown in figure below. The vertical asymptote is $x = 0$ (y-axis).



21.13. LOGARITHM BASE PROPERTIES

Before we proceed ahead for logarithm properties, we need to revise the law of exponents, so that we can compare the properties.

For exponents, the laws are

- Product rule: $a^m.a^n = a^{m+n}$
- Quotient rule: $a^m/a^n = a^{m-n}$
- Power of a power: $(a^m)^n = a^{mn}$

Now let us learn the properties of logarithm functions.

Product Property

If a, m and n are positive integers and $a \neq 1$, then

$\log_a(mn) = \log_a m + \log_a n$

Thus, the log of two number m and n, with base 'a' is equal to the sum of log m and log n with the same base.

Example 11. $log_3 (9.25)$

Solution. $\log_3 (9) + \log_3 (27)$

$= \log_3 (3^2) + \log_3 (3^3)$

$= 2 + 3$(By property: $\log_b bx = x$)

$= 5$

21.14. QUOTIENT PROPERTY

If m, n and a are positive integers and $a \neq 1$, then

$\log_a(m/n) = \log_a m - \log_a n$

In the above expression, the logarithm of a quotient of two positive numbers m and n results in a difference of log of m and log n with the same base 'a'.

Example 12. $log_2 (21/8)$.

Solution. $\log_2 (21/8) = \log_2 21 - \log_2 8$

21.15. POWER RULE

If a and m are positive numbers, $a \neq 1$ and n is a real number, then

$$\log_a m^n = n \log_a m$$

The above property defines that logarithm of a positive number m to the power n is equal to the product of n and log of m.

21.16. CHANGE OF BASE RULE

If m, n and p are positive numbers, $n \neq 1$, $p \neq 1$, then:

$$\log_n m = \log_p m / \log_p n$$

Example 13. $log_2\ 10 = log_p\ 10 / log_p\ 2$

Solution. Reciprocal rule

If m and n are the positive numbers other than 1, then;

$$\log_n m = 1/\log_m n$$

21.17. LOGARITHMIC FUNCTIONS AND ITS TYPES

The logarithm is defined as a quantity that represents the powe in which the base (fixed number) is raised to produce a given number. The general representation of the logarithmic function is:

In general, the two different types of logarithmic functions are

- Common Logarithmic Function
- Natural Logarithmic Function

In common logarithmic function, the base of the logarithmic function is 10. Log_{10} or log represents this function.

In natural logarithmic function, the base of the logarithmic function is e. Log_e or ln represents this function.

21.18. VALUE OF LOG 1 TO 10 FOR LOG BASE 10

The value of log 1 to 10 (common logarithm $\log_{10} x$)

Common Logarithm to a Number (log_{10} x)	***Log Value***
Log 1	0
Log 2	0.3010
Log 3	0.4771
Log 4	0.6020
Log 5	0.6989
Log 6	0.7781
Log 7	0.8450
Log 8	0.9030
Log 9	0.9542

Example 14. *Find the value of* $log_{10}(75/16) - 2\,log10(5/9) + log(32/243)$.

Solution. We know that $\log_{\mathrm{a}} m^n = n \log_{\mathrm{a}} m$

$$\log_a (p/q) = \log_a p - \log_a q$$

$$\log_a (mn) = \log_a m + \log_a n$$

Using these logarithmic rules, we have

$$\log_{10} (75/16) - 2\log_{10}(5/9) + \log(32/243)$$
$$= \log_{10} (75/16) - \log_{10}(5/9)^2 + \log(32/243)$$
$$= \log_{10} (75/16) - \log_{10}(25/81) + \log(32/243)$$
$$= \log_{10} [(75/16) \times (32/243)] - \log_{10}(25/81)$$
$$= \log_{10} (50/81) - \log_{10}(25/81)$$
$$= \log_{10} [(50/81)/(25/81)]$$
$$= \log_{10} [(50/81) \times (25/81)]$$
$$= \log_{10} [(50/81) \times (81/25)]$$
$$= \log_{10} 2 = 0.3010$$

$\therefore \quad \log_{10}(75/16) - 2\log_{10}(5/9) + \log(32/243) = 0.3010$

Example 15. *Evaluate* $log_{10}(124416)$.

Solution.
$$\log_{10} (124416) = \log_{10} (512 \times 243)$$
$$= \log_{10} (29 \times 35)$$
$$= \log_{10} 29 + \log_{10} 35$$
$$= 9 \log_{10} 2 + 5 \log_{10} 3$$
$$= 9 \times 0.3010 + 5 \times 0.4771$$
$$= 2.709 + 2.3855$$
$$= 5.0945$$

$\therefore \quad \log_{10}(122416) = 5.0945$

21.19. CHANGE OF BASE OF FORMULA

The change of base formula helps to rewrite the logarithm in terms of another base log. Change of base formula is used in the evaluation of log and have another base 10.

$$\log_b x = \frac{\log_d x}{\log_d x}$$

Example 16. *Simplify $\log_{32} 16$.*

Solution. Given $\log_{32} 16$

Using change of base formula,

$$\log_{32} 16 = \frac{\log_{10} 16}{\log_{10} 32} = \frac{\log_{10} 2^4}{\log_{10} 2^5} = \frac{4 \log_{10} 2}{5 \log_{10} 2} = \frac{4}{5}$$

21.20. LOGARITHMS TO BASES OTHER THAN 10

Let $b > 0$ and $b \neq 1$. Then x is the logarithm of a to the base b written $x = \log_b a$, if and only if $b^x = a$.

For example, because $3^5 = 243$, you can write $5 = \log_3 243$. This is read "5 is the logarithm of 243 with base 3" or "5 is log 243 to the base 3" or "5 is the log base 3 of 243". At the right are some other power of 3 and the related logs of the base of 3

Exponential Form	***Logarithmic Form***
$3^4 = 81$	$\log_3 81 = 4$
$3^3 = 27$	$\log_3 27 = 3$
$3^2 = 9$	$\log_3 9 = 2$
$3^1 = 3$	$\log_3 3 = 1$
$3^{0.5} = 81$	$\log_3 \sqrt{3} = 0.5$
$3^0 = 1$	$\log_3 1 = 0$
$3^{-1} = 81$	$\log_3 \left(\frac{1}{3}\right) = -1$
$3^{-2} = 81$	$\log_3 \left(\frac{1}{9}\right) = 4$
$3^y = 81$	$\log_3 x = y$

21.21. SOLVING LOGARITHMIC EQUATIONS

To solve a logarithmic equation, it often helps to use the definition of logarithm to rewrite the equation in exponential form

Example 17. *Solve for h :*$\log_4 = \frac{3}{2}$.

Solution. $4^{\frac{3}{2}} = h$ definition of logarithm

$8 = h$ Rational Exponent Theorem

EXERCISE

1. Graphing exponential functions for $b < 1$. Graph the function $f(x) = \left(\frac{2}{5}\right)^x$
2. Graphing exponential functions with base *e*. Graph the function $f(x) = 3 + e^x$.
3. Graphing the exponential functions. Sketch the graph of each function. State the domain, range and asymptote.

   (*a*) $f(x) = e^{-x}$ (*b*) $g(x) = 3e^{0.5x}$
4. Use a calculator to evaluate the function $f(x) = \log x$ at each value of *x*.

   (*a*) $x = 10$ (*b*) $x = \frac{1}{3}$

   (*c*) $x = -2$

PERIOD-VI

M12CH22

TOPIC 22

Differentiation and Integration

22.1. DIFFERENCE QUOTIENT FORMULA

In single-variable calculus, the difference quotient is usually the name for the expression, which taken to the limit when h approaches 0, gives the derivative of the function f. Difference Quotient Formula is used to find the slope of the line that passes through two points. It is also used in the definition of the derivative.

$$\text{Difference Quotient Formula} = \frac{f(x+h) - f(x)}{h}$$

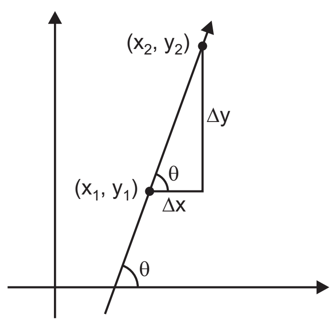
Example 1. *Find the difference of the function $f(x) = 3x - 5$.*

Solution. Using the difference quotient formula

$$\begin{aligned}\text{Difference quotient of } f(x) &= [f(x + h) - f(x)]/h \\ &= [3(x + h) - 5) - (3x - 5)]/h \\ &= [3x + 3h - 5 - 3x + 5]/h = [3h]/h = 3\end{aligned}$$

22.2. WHAT IS SLOPE?

The slope of a line is defined as the change in y-coordinate with respect to the change in x-coordinate of that line. The net change in y-coordinate is Δy, while the net change in the x-coordinate is Δx. So, the change in y-coordinate with respect to the change in x-coordinate can be written as,



$$m = \Delta y/\Delta x$$

where m is the slope

Note that $\tan\theta = \Delta y/\Delta x$

We also refer this $\tan\theta$ to be the slope of the line.

22.3. SLOPE OF A LINE

The slope of the line is the ratio of the rise to the run, or rise divided by the run. It describes the steepness of line in the coordinate plane. Calculating the slope of a line is similar to finding the slope between two different points. In general, to find the slope of a line, we need to have the values of any two different coordinates on the line.

Example 2. *Given a line with the equation $2y = 8x + 9$, find its slope.*

Solution. We know that the general formula of the slope is given as, $y = mx + b$

Hence, we try to bring the equation to this form. We make the coefficient of $y = 1$, and hence we get,

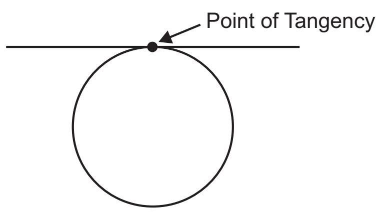
$$y = 4x + 4.5$$

Clearly, the coefficient of x is found to be 4. Hence, our slope will be same as the coefficient of x.

The slope is 4.

22.4. WHAT IS TANGENT LINE?

The **tangent line** of a curve at a given point is a line that just touches the curve (function) at that point. The tangent line in calculus may touch the curve at any other point(s) and it also may cross the graph at some other point(s) as weill. The point at which the tangent is drawn is known as the "point of tangency". We can see the tangent of circle drawn here.



22.5. CONCEPT OF LIMIT

Let $f(x)$ be a function defined for all x in the nbd of 'a' except possibly at 'a'. Then, l is said to be the limiting value of $f(x)$ as x tends to a. If the numercial difference between $f(x)$ and l can be made as small as we please by taking x sufficiently close to 'a' but not equal to 'a'.

We write this as: $\lim_{x \to a} f(x) = l$

Definition. Let f be a function defined in a nbd of a except possible at a. Then, a real number l is said to be a limit of f as tends to a if for any $\varepsilon > 0$, however small, there exists $\delta > 0$ (depending upon ε) such that:

$| f(x) - l | < \varepsilon$, whenever $0 < | x - a | < \delta$.

i.e. $l - \varepsilon < f(x) < l + \varepsilon$, whenever $x \in (a - \delta, a) \cup (a, a + \delta)$

We write $\lim_{x \to a} f(x) = l$

Remark. The limit of f at a, if exists, will continue to exist and be the same if we change the value of f at a only.

Left Hand Limit of a Function

Let $f(x)$ be a function of x.

If $f(x)$ approaches to l as $x \to a^-$, then l is called the left hand limit of the function and we write it as:

$$\lim_{x \to a^-} f(x) = l$$

Right Hand Limit of a Function

Let $f(x)$ be a function of x if $f(x)$ approaches to l as $x \to a^+$, then l is called the right hand limit of the function and we write it as:

$$\lim_{x \to a^+} f(x) = l$$

Remark.

(*i*) $\lim_{x \to a} f(x)$ exists if and only if

$\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist and are equal i.e.,

$\lim_{x \to a} f(x)$ exists iff $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

(*ii*) If $x \to a^-$ and $x = a - h$, then we have $h = (a - x)$ and as $x \to a^-$, the difference $a - x$ (= h) is positive and is close to zero.

Fundamental Theorems on Limits

Here we list some of the fundamental results involving limits of functions. The proofs of these are beyond the scope of this book.

(*i*) If $f(x) = k$, a constant function, then $\lim_{x \to a} f(x) = k$

(*ii*) $\lim_{x \to a}[k\, f(x)] = k \lim_{x \to a} f(x)$ where : k is a consant

(*iii*) $\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

(*iv*) $\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

(*v*) $\lim_{x \to a}[f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

(*vi*) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$

(*vii*) If $f(x) \leq g(x)$ for all x, then: $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$

(*viii*) $\lim_{x \to a}\left(\dfrac{1}{f}\right)(x) = \lim_{x \to a} \dfrac{1}{f(x)} = \dfrac{1}{\lim_{x \to a} f(x)}$, provided $\lim_{x \to a} f(x) \neq 0$

(*ix*) $\lim_{x \to a}[f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$

Example 3. *Evaluate the following limits:*

(*i*) $\lim_{x \to 2} \dfrac{x^3 - 8}{x^2 - 4}$ (*ii*) $\lim_{x \to 2} \dfrac{x^2 - 4}{x + 2}$ (*iii*) $\lim_{x \to 2} \dfrac{x^2 - 5x + 6}{x^2 - 4}$

(*iv*) $\lim_{x \to -1} \dfrac{x^3 + 1}{x + 1}$ (*v*) $\lim_{x \to 0} \dfrac{2x^2 + 3x + 4}{x^2 + 3x + 2}$

Solution. (*i*) When $x = 2$, the given expression assumes the indeterminate form $\left(\dfrac{0}{0}\right)$.

$$\therefore \quad \lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4} = \lim_{x \to 2} \frac{x^3 - 2^3}{x^2 - 2^2} \qquad \left(\text{Form } \frac{0}{0}\right)$$

$$= \lim_{x \to 2} \frac{(x - 2)(x^2 + 2x + 4)}{(x - 2)(x + 2)}$$

$$[\because \quad a^3 - b^3 = (a - b)(a^2 + ab + b^2)]$$

$$= \lim_{x \to 2} \frac{x^2 + 2x + 4}{x + 2} = \frac{(2)^2 + 2(2) + 4}{2 + 2} = \frac{12}{4} = 3$$

(*ii*) We have, $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x + 2} = \dfrac{(2)^2 - 4}{2 + 2} = \dfrac{4 - 4}{4} = \dfrac{0}{4} = 0$

(*iii*) When $x = 2$, the given expression assumes the indeterminate form $\left(\dfrac{0}{0}\right)$

$$\therefore \quad \lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - 4} = \lim_{x \to 2} \frac{(x - 2)(x - 3)}{(x + 2)(x - 2)} \qquad \left(\text{Form } \frac{0}{0}\right)$$

$$= \lim_{x \to 2} \frac{x - 3}{x + 2} = \frac{2 - 3}{2 + 2} = -\frac{1}{4}$$

(*iv*) When $x = -1$, the given expression assumes the indeterminate form $\left(\dfrac{0}{0}\right)$

$$\therefore \quad \lim_{x \to -1} \left(\frac{x^3 + 1}{x + 1}\right) = \lim_{x \to -1} \frac{x^3 + 1^3}{x + 1} \qquad \left(\text{Form } \frac{0}{0}\right)$$

$[\because \quad a^3 - b^3 = (a - b)(a^2 + ab + b^2)]$

$$= \lim_{x \to -1} \frac{(x + 1)(x^2 - x + 1)}{x + 1} = \lim_{x \to -1} (x^2 - x + 1)$$

$$= (-1)^2 - (-1) + 1 = 1 + 1 + 1 = 3$$

(*v*) We have,

$$\lim_{x \to 0} \frac{2x^2 + 3x + 4}{x^2 + 3x + 2} = \frac{0 + 0 + 4}{0 + 0 + 2} = \frac{4}{2} = 2.$$

22.6. DIFFERENTIATION

In calculus, differentiation is one of the two important concepts apart from integration. Differentiation is a method of finding the derivative of a function. Differentiation is a process, in Maths, where we find the instantaneous rate of change in function based on one of its variables. The most common example is the rate change of displacement with respect to time, called velocity. The opposite of finding a derivative is anti-differentiation.

If x is a variable and y is another variable, then the rate of change of x with respect to y is given by dy/dx. This is the general expression of derivaive of a function and is represented as $f'(x) = dy/dx$, where $y = f(x)$ is any function.

22.7. CONCEPT OF DIFFERENTIATION

1. **Product rule of Differentiation:**

   Definition: If u and v are two differentiable functions of x, then

$$\frac{d}{dx}(uv) = u \cdot \frac{d}{dx} + v \cdot \frac{du}{dx}$$

2. **Quotient rule of Differentiation:**

   Definition: If u and v are differentiable of x, then

Then $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{du}{dx}}{v^2}$$

3. **Derivative of function of a function (Chain rule):**

   Definition: If $y = f(u)$ and $u = g(x)$ are differentiable function of u and x respecitvely.

Then $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Derivative of Algebraic Functions:

If $y = x^n$, then $$\frac{d}{dx}x^n = nx^{n-1}$$

Derivative of Algebraic Functions:

If $y = x$, then $$\frac{d}{dx}(c) = 0$$

Example 4. *Differentiate the following functions w.r.t. x.*

(*i*) $ax^3 + bx^2 + cx + d$ (*ii*) $\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right)$

(*iii*) $(2x + 3)^2$ (*iv*) $\left(x + \frac{1}{x}\right)\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$

Solution. (*i*) Let $y = ax^3 + bx^2 + cx + d$

$$\therefore \quad \frac{dy}{dx} = 3ax^2 + 2bx + c.1 + d \qquad \left[\because \frac{d}{dx}(x^n = nx^{n-1})\right]$$

$\Rightarrow \qquad \frac{dy}{dx} = 3ax^2 + 2bx + c$

(*ii*) $\qquad y = \left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right)$

$= x^3 + \frac{1}{x} + x + \frac{1}{x^3}$

$\therefore \qquad \frac{dy}{dx} = 3x^2 - 1x^{-2} + 1 - 3x^{-4} = 3x^2 - \frac{1}{x^2} + 1 - \frac{3}{x^4}$

$\left[\because \frac{d}{dx}(x^n = nx^{n-1})\right]$

(*iii*) Let $\qquad y = (2x + 3)^2$

$= 4x^2 + 9 + 2(2x)(3) \quad [\because (a + b)^2 = a^2 + b^2 + 2ab]$

$= 4x^2 + 9 + 12x$

$\therefore \qquad \frac{dy}{dx} = 8x + 12 + 0 = 8x + 12 \qquad \left[\because \frac{d}{dx}(x^n = nx^{n-1})\right]$

(*iv*) Let $\qquad y = \left(x + \frac{1}{x}\right)\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$

$= x\sqrt{x} + \frac{\sqrt{x}}{x} + \frac{x}{\sqrt{x}} + \frac{1}{x\sqrt{x}}$

$= x^{3/2} + \frac{1}{\sqrt{x}} + \sqrt{x} + \frac{1}{x^{3/2}}$

$\Rightarrow \qquad y = x^{3/2} + x^{-1/2} + x^{1/2} + x^{-3/2}$

$\therefore \qquad \frac{dy}{dx} = \frac{3}{2}x^{\frac{3}{2}-1} + \left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1} + \frac{1}{2}x^{\frac{1}{2}-1} + \left(-\frac{3}{2}\right)x^{-\frac{3}{2}-1}$

$\left[\because \frac{d}{dx}(x^n = nx^{n-1})\right]$

$= \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-3/2} + \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{-5/2}$

$= \frac{3}{2}\sqrt{x} - \frac{1}{2x^{3/2}} + \frac{1}{2x^{1/2}} - \frac{3}{2x^{5/2}}$

$$= \frac{3}{2}\sqrt{x} - \frac{1}{2x\sqrt{x}} + \frac{1}{2\sqrt{x}} - \frac{3}{2x^2\sqrt{x}}$$

$$= \frac{1}{2}\left(3\sqrt{x} - \frac{1}{x\sqrt{x}} + \frac{1}{\sqrt{x}} - \frac{3}{x^2\sqrt{x}}\right)$$

22.8. INTEGRATION

Integration is a method of adding or summing up the parts to find the whole. It is a reverse process of differentiation, where we reduce the functions into parts. This method is used to find the summation under a vast scale. Calculation of small addition problems in an easy task which we can do manually or by usiung calculations as well. But for big addition problems, where the limits could reach to even infinity. Integration methods are used. Integration and differentiation both are important parts of calculus. The concept level of these topics is very high. Hence, it is introduced to us at higher secondary classes and thein in engineering or higher education. To get an in-depth knowledge of integration, read the complete article here.

Important Extension Formulae of Standard Integral Forms

(*i*) $$\int x^n \cdot dx = \frac{x^{n+1}}{n+1} + c; \ (n \neq -1)$$

$$\Rightarrow \int (ax+b)^n \cdot dx = \frac{(ax+b)^{n+1}}{(n+1)\cdot a} + c; (n \neq -1)$$

(*ii*) $$\int \frac{1}{x} \cdot dx = \log|x| + c$$

$$\Rightarrow \int \frac{1}{(ax+b)} \cdot dx = \frac{\log|ax+b|}{a} + c$$

(*iii*) $$\int a^x \cdot dx = \frac{a^x}{\log a} + c; \ (a > 0, \ a \neq 1)$$

$$\Rightarrow \int a^{mx+b} \cdot dx = \frac{a^{mx+b}}{m \cdot \log a} + c; \qquad (a > 0; \ a \neq 1)$$

(iv) $\int e^x \cdot dx = e^x + c$

$\Rightarrow \int e^{mx+b} \cdot dx = \dfrac{e^{mx+b}}{m} + c$

(v) $\int \cos x \cdot dx = \sin x + c$

$\Rightarrow \int \cos (ax + b)dx = \dfrac{\sin (ax + b)}{a} + c$

(vi) $\int \sin x\, dx = - \cos x + c$

$\Rightarrow \int \sin (ax + b)dx = -\dfrac{\cos (ax + b)}{a} + c$

(vii) $\int \sec^2 x\, dx = \tan x + c$

$\Rightarrow \int \sec^2 (ax + b)dx = \dfrac{\tan (ax + b)}{a} + c$

(viii) $\int \text{cosec}^2\, x + dx = - \cot x + c$

$\Rightarrow \int \text{cosec}^2 (ax + b)dx = \int \dfrac{- \cot (ax + b)}{a} + c$

(ix) $\int \sec x \tan x\, dx = \sec x + c$

$\Rightarrow \int \sec (ax + b) \tan (ax + b)\, dx = \dfrac{\sec(ax + b)}{a} + c$

(x) $\int \text{cosec}\, x + \cot x\, dx = - \text{cosec}\, x + c$

$\Rightarrow \int \text{cosec}\, (ax + b) + \cot(ax + b)dx = \int -\dfrac{\text{cosec}\, (ax + b)}{a} + c$

Example 5. *Evaluate the following integrals*

(i) $\int (1 - x)\sqrt{x}\, dx$ (ii) $\int \sqrt[3]{x - dx}$

(iii) $\int a^x \cdot e^x\, dx$ (iv) $\int (x^2 - 2x + 4)^2 \cdot dx$

(v) $\int 9^{x+2}\, dx$ (iv) $3^{2\log_3 x}\, dx$

Solution. (i) $\int (1 - x)\sqrt{x}dx = \int (\sqrt{x} - x\sqrt{x})dx = \int x^{1/2} - \int x^{3/2}dx$

$\left[\because \int x^n\, dx = \dfrac{x^{n+1}}{n+1} + c\right]$

(*ii*) $\int \sqrt[3]{x}\, dx = \int x^{1/3}\, dx = \dfrac{x^{1/3+1}}{\frac{1}{3}+1} + c$ $\left[\because \int x^n\, dx = \dfrac{x^{n+1}}{n+1} + c\right]$

(*iii*) $\int a^x \cdot e^x\, dx = \int (ae)^x\, dx = \dfrac{(ae)^x}{\log (ae)} + c$

$$\left[\because \int a^x dx = \frac{a^x}{\log a} + c\right]$$

(*iv*) $\int (x^2 - 2x + 4)^2\, dx$

$$[\because \quad (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca]$$

$$= \int (x^4 + 4x^2 + 16 - 4x^3 - 16x + 8x^2)dx$$

$$= \int (x^4 - 4x^3 + 12x^2 - 16x + 16)dx$$

$$= \int x^4\, dx - 4\int x^2\, dx - 16\int x\, dx + 16\int dx$$

$$\left[\because \int x^n\, dx = \frac{x^{n+1}}{n+1} + c\right]$$

$$= \frac{x^5}{5} - \frac{4x^4}{4} + 12\frac{x^3}{3} - 16\frac{x^2}{2} + 16x + c$$

$$= \frac{1}{5}x^5 - x^4 + 4x^3 - 8x^2 + 16x + c$$

(*v*) $\int 9^{x+2}\, dx = \int 9^x \cdot 9^2\, dx$

$$= \int 81.9^x dx = 81\int 9^x dx$$

$$= 81\left(\frac{9^x}{\log 9}\right) + c \qquad \left[\because \int a^x\, dx = \frac{a^x}{\log a} + c\right]$$

(*vi*) $\int 3^{2\log_3 x}\, dx = \int 3^{\log_3 x^2}\, dx$ $[\because \quad m \log n = \log n^m]$

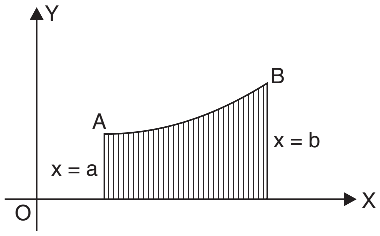
$$= \int x^2\, dx = \frac{x^{2+1}}{2+1} + c \qquad [\because \quad a^{\log_a x} = x]$$

$$= \frac{1}{3}x^3 + c$$

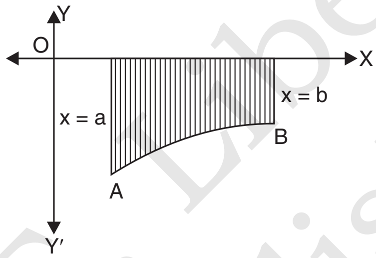
22.9. AREA UNDER THE CURVE

1. The area under the curve $y = f(x)$, above x-axis between the ordinates $x = a$ and $x = b$ is given by

$$\int_a^b y.dx = \int_a^b f(x)\,dx$$



(a)



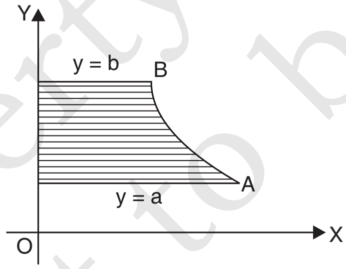
(b)

2. The area bounded by the curve $y = f(x)$, below x-axis between the ordinates $x = a$ and $x = b$ is given by

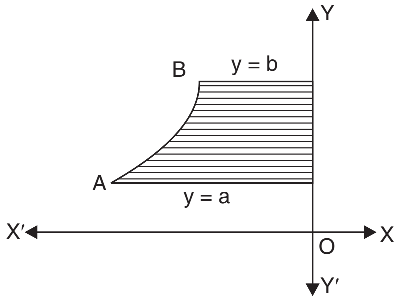
$$\int_a^b -y\,dx = -\int_a^b y\,dx = -\int_a^b f(x)\,dx$$

3. The area bounded by the curve $x = f(y)$, y-axis between the abscissae $y = a$ and $y = b$ is given by

$$\int_a^b x\,dy = \int_a^b f(y)\,dy$$



(a)



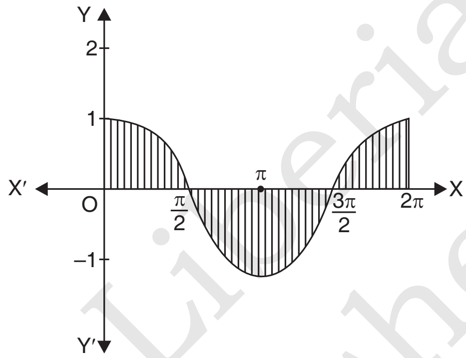
(b)

4. The area bounded by the curve $x = f(y)$, y-axis between the abscissae $y = a$ and $y = b$ is given by

$$\int_a^b -x\,dy = \int_a^b f(y)\,dy$$

Example 6. *Find the area bounded by the curve y = cos x, x-axis and the ordinates x = 0 and x = 2π.*

Solution. The equation of the given curve is $y = \cos x$



Now $\cos x > 0$ when $x \in \left(0, \frac{\pi}{2}\right)$

$\cos x < 0$ when $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

$\cos x > 0$ when $x \in \left(\frac{3\pi}{2}, 2\pi\right)$

⇒ $f(x)$ changes sign in the given interval.

Table of values of y corresponding to values of x from 0 to 2π is given by

x	0	π/2	π	3π/2	2π
y	1	0	– 1	0	1

By joining these points with a free hand we obtain a rough sketch of the curve as shown in the figure,

∴ Required area

= Area of shaded region

$$= \int_0^{2\pi} |y| \cdot dx = \int_0^{\pi/2} |y| \cdot dx + \int_{\pi/2}^{3\pi/2} |y| \cdot dx + \int_{3\pi/2}^{2\pi} |y| \cdot dx$$

$$= \int_0^{\pi/2} |\cos x\, dx + \int_{\pi/2}^{3\pi/2} -\cos x\, dx + \int_{3\pi/2}^{2\pi} \cos x\, dx$$

$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{3\pi/2} + [\sin x]_{3\pi/2}^{2\pi}$$

$$= \left[\sin\frac{\pi}{2} - \sin 0\right] - \left[\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right] + \left[\sin 2\pi - \sin\frac{3\pi}{2}\right]$$

= (1 – 0) – (– 1 – 1) + [0 – (– 1)] = 1 + 2 + 1

= 4 sq. units

22.10. INDEFINITE INTEGRAL

Definition. Let $f(x)$ *be a function. Then the family of all its primitives (or antiderivatives) is called the indefinite integral of f(x) and is denoted by* $\int f(x)\,dx$.

The symbol $\int f(x)\,dx$ is read as the indefinite integral of $f(x)$ with respect to x.

Thus, $$\frac{d}{dx}(\phi(x) + C) = f(x) \quad \Leftrightarrow \quad \int f(x)\,dx = \phi(x) + C$$

where $\phi(x)$ is primitive of $f(x)$ and C is an arbitrary constant known as the *constant of integration.*

Here, $\int$ is the integral sign, $f(x)$ is the integrand, x is the variable of integration and dx is the element of integration or differential of x.

22.11. INDEFINITE INTEGRALS OF TRIGONOMETRIC FUNCTIONS

Example 7. *Evaluate the following integrals:*

(*i*) $\int \tan^2 x \cdot dx$ (*ii*) $\int \sqrt{1 - \sin 2x} \cdot dx$

(*iii*) $\int (\sin x + \cos x) \cdot dx$ (*iv*) $\int \operatorname{cosec} x\,(\operatorname{cosec} x + \cot x) \cdot dx$

(*v*) $\int \frac{1 - \sin x}{\cos^2 x} \cdot dx$

Solution. (*i*) $$\int \tan^2 x\,dx = \int (\sec^2 x - 1)dx$$

$$[\because \quad \sec^2 A - \tan^2 A = 1]$$

$$= \int \sec^2 x \cdot 1 - \int 1 \cdot dx = \tan x - x + c$$

(*ii*) $$\int \sqrt{1 - \sin 2x} \cdot dx = \int [(\cos^2 x + \sin^2 x) - 2 \sin x \cos x]^{1/2} \cdot dx$$

$$[\because \quad \cos^2 A + \sin^2 A = 1,\ \sin 2A = 2 \sin A \cos A]$$

$$= \int [(\cos x - \sin x)^2]^{1/2} \cdot dx$$

$$= \int (\cos x - \sin x)\cdot dx$$

$$= \int \cos x\, dx - \int \sin x\, dx$$

$$= \sin x - (-\cos x) + c$$

$$= \sin x + \cos x + c$$

(*iii*) $\int (\sin x + \cos x)\cdot dx = \int \sin x\, dx + \int \cos x\, dx$

$$= -\cos x + \sin x + c$$

(*iv*) $\int \text{cosec } x(\text{cosec } x + \cot x)\cdot dx$

$$= \int \text{cosec}^2 x\, dx + \int \text{cosec } x \cot x \cdot dx$$

$$= -\cot x - \text{cosec } x + c$$

(*v*) $\int \frac{1 - \sin x}{\cos^2 x} dx = \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right)\cdot dx$

$$= \int (\sec^2 x - \sec x \tan x)\cdot dx$$

$$= \int \sec^2 x \cdot dx - \int \sec x \tan x \cdot dx$$

$$= \tan x - \sec x + c$$

EXERCISE

1. Find the difference quotient of function f is defined by $f(x) = 2x + 5$.

2. Evaluate the following limits:

(*i*) $\lim_{x\to 1/2} \frac{8x^3 - 1}{16x^4 - 1}$ (*ii*) $\lim_{x\to \pi/2} \left(\frac{\sin^2 x + \cos 2x}{x}\right)$

(*iii*) $\lim_{x\to 1} \left(\frac{2}{1 - x^2} + \frac{1}{x - 1}\right)$

3. Evaluate the following limits:

(*i*) $\lim_{x\to 2} \left(\frac{x}{x - 2} - \frac{4}{x^2 - 2x}\right)$ (*ii*) $\lim_{x\to 3} (x^2 - 9)\left[\frac{1}{x + 3} + \frac{1}{x - 3}\right]$

(*iii*) $\lim_{x\to 0} \frac{(1 + x)^6 - 1}{(1 + x)^2 - 1}$ (*iv*) $\lim_{x\to 3} \left(\frac{1}{x - 3} - \frac{2}{x^2 - 4x + 3}\right)$

(*v*) $\lim_{x\to -3} \frac{x^3 + 4x^2 + 4x + 3}{x^2 + 2x - 3}$

4. Differentiable the following functions w.r.t. x.

(*i*) $(x^2 + 4x + 5)$

(*ii*) $(2x + 1)^{1/3} (x + 1)$

(*iii*) $(x^2 + 1) (x^2 + x + 4)$

(*iv*) $(3x + 8)^{7/3} + (x + 7)^{-3}$

5. Differentiable the following functions w.r.t. x.

(*i*) $\dfrac{2x+3}{x^2-5}$

(*ii*) $\dfrac{3x+2}{(x+5)(2x+1)+3}$

(*iii*) $\sqrt{\dfrac{1+x}{1-x}}$

(*iv*) $\dfrac{x^4+1}{x^2+1}$

6. Evaluate the following integrals:

(*i*) $\int \sqrt{ax+b}\, dx$

(*ii*) $\int (4x^3 - 4x^{-5}) dx$

(*iii*) $\int \dfrac{x^3 - x^2 + x - 1}{x-1} dx$

(*iv*) $\int \dfrac{x^2}{1+x^2} dx$

(*v*) $\int (x-3)^2 \cdot \sqrt{x}\, dx$

7. Find the area of the curve $y = x^2 - 2x$ lying below the x-axis.